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(Like a rolling) Stone to CFTs: A study of dynamics

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1 Introduction

When we discuss dynamics a few natural axioms appear. Doing nothing should leave us in place. Moving from 0 to s and then from s to $s + t$ should be the same as moving from 0 to $s + t$. For reversible dynamics every time-step has an inverse. These natural axioms precisely define a group homomorphism

$$U : (\mathbb{R}, +) \rightarrow \text{Aut}(X), \quad t \mapsto U_t.$$

Where X is an object in some appropriate category.

The properties we expected are now encoded into the structure of the group homomorphism i.e., $U_0 = \text{id}_X$, $U_{s+t} = U_s \circ U_t$, and $U_{-t} = U_t^{-1}$.

In the second section we discuss time reversible dynamics for Hilbert spaces $\mathcal{H}$. Where we have $\text{Aut}(\mathcal{H}) = \mathcal{U}(\mathcal{H})$. Here Stone's theorem identifies such evolutions as unitary exponentials $U_t = e^{itA}$ for a unique self-adjoint generator A .

In the third section we discuss irreversible dynamics which move only forward in time. There we drop invertibility and obtain a semigroup morphism instead

$$T : ([0, \infty), +) \rightarrow \text{End}(X), \quad t \mapsto T_t,$$

with $T_0 = \text{id}_X$ and $T_{s+t} = T_s \circ T_t$.

In particular Section 3 examines the irreversible case and proves the Hille–Yosida theorem.

In this text our convention is to work with Hilbert spaces over $\mathbb{C}$. Additionally we use i to refer to $\sqrt{-1}$.

2 Reversible dynamics: Stone's theorem

In the context of quantum mechanics and Stone's theorem we consider **Hilb** the category whose objects are Hilbert spaces and morphisms are unitary morphisms.

Stone's theorem gives us a particularly nice description of dynamics in this setting via exponentials. In particular it says that particular continuous one-parameter family of unitary groups $U(t)$ on some Hilbert space $\mathcal{H}$ can be represented as $U(t) = e^{itA}$ for a unique self adjoint A .

It is pedagogically sound to first see the special case when $\mathcal{H} \in \mathbf{Hilb}$ is finite dimensional.

Before the finite-dimensional case we fix conventions and recall two definitions.

Definition 2.1 (Hilbert space). *A complex Hilbert space is a complex vector space H with an inner product $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$ that is linear in the first variable, conjugate symmetric $\langle x, y \rangle = \overline{\langle y, x \rangle}$, positive definite $\langle x, x \rangle > 0$ for $x \neq 0$, and complete in the norm $\|x\| = \sqrt{\langle x, x \rangle}$.*

Definition 2.2 (Matrix exponential). *For $B \in M_n(\mathbb{C})$ define*

$$e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}, \quad e^{tB} = \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \quad (t \in \mathbb{R}).$$

In finite dimensions these series converge absolutely in any operator norm.

Lemma 2.3 (Basic properties used later). *Let $B, C \in M_n(\mathbb{C})$. Then:*

1. $e^0 = I$ and $(e^{tB})^{-1} = e^{-tB}$.
2. $e^{sB}e^{tB} = e^{(s+t)B}$ for $s, t \in \mathbb{R}$.
3. If $BC = CB$ then $e^B e^C = e^{B+C}$.
4. If A is self-adjoint then e^{itA} is unitary, $(e^{itA})^* = e^{-itA}$, and $\|e^{itA}\| = 1$.
5. If $B = SDS^{-1}$ with D diagonal then $e^B = Se^D S^{-1}$ and $e^{tB} = Se^{tD} S^{-1}$.

2.1 Finite dimension case

We begin with the finite-dimensional case, where everything reduces to Hermitian matrices and the usual matrix exponential.

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Without loss of generality we may consider $\mathcal{H} = \mathbb{C}^n$. This is valid since any n -dimensional complex Hilbert space is unitarily isomorphic to $\mathbb{C}^n$.

Let us begin by seeing what happens in the 1-dim case, i.e. when we are looking at just $\varphi : \mathbb{R} \rightarrow \text{Aut}((\mathbb{C}, +)) \cong (\mathbb{C}^\times, \cdot)$. We want a continuous map $\phi : \mathbb{R} \rightarrow \mathbb{C}^\times$ such that $\phi(0) = 1, \phi(t+s) = \phi(t)\phi(s)$.

Additionally our setting is **Hilb** with unitary morphisms. Therefore we require $|\phi(t)| = 1$.

Lemma 2.4. *A continuous additive function $f : \mathbb{R} \rightarrow \mathbb{R}$ is linear.*

Proof. Linearity on integers and rationals are immediate. To extend to reals use density of rationals and continuity of f . $\square$

Lemma 2.5. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be continuous, $f(0) = 1, |f(t)| = 1, f(t+s) = f(t)f(s)$ then there exists a unique $\lambda \in \mathbb{R}$ with $f(t) = e^{i\lambda t}$ for all $t \in \mathbb{R}$.*

Proof. Since f is continuous, $f(\mathbb{R}) \subset S^1$ is connected and contains 1. The covering map $\exp : \mathbb{R} \rightarrow S^1, x \mapsto e^{ix}$, gives us a continuous lift $\theta : \mathbb{R} \rightarrow \mathbb{R}$ with $f = e^{i\theta}$ and $\theta(0) = 0$.

The functional equation yields

$$e^{i\theta(t+s)} = e^{i(\theta(t)+\theta(s))},$$

so $\theta(t+s) - \theta(t) - \theta(s) \in 2\pi\mathbb{Z}$.

The left-hand side is continuous in (t, s) and vanishes at $(0, 0)$. Hence it is identically zero. Therefore, θ is additive. By Lemma 2.4, $\theta(t) = \lambda t$ for some $\lambda \in \mathbb{R}$, so $f(t) = e^{i\lambda t}$.

Uniqueness: if $e^{i\lambda t} = e^{i\mu t}$ for all t , then $(\lambda - \mu)t \in 2\pi\mathbb{Z}$ for all t , forcing $\lambda = \mu$. $\square$

The above lemma characterizes our one dimensional case by demonstrating that the exponential is a unique solution to the group homomorphism and norm preserving conditions. Extending this to finite dimensions will use the key lemma 2.5 as its core.

Proposition 2.6 (Stone's theorem in finite dimensions). *A continuous group homomorphism $\varphi : \mathbb{R} \rightarrow \text{Aut}_{\mathbf{Hilb}}(\mathbb{C}^n)$ is uniquely determined by a self adjoint operator $H \in M_n(\mathbb{C})$ in the following sense:*

$$\varphi(t) = e^{itH}$$

for all $t \in \mathbb{R}$.

Proof. Each $\varphi(t)$ is unitary with inverse $\varphi(t)^*$. We also have that $\varphi(t)$ is normal since $\varphi(t)\varphi(t)^* = \varphi(t)^*\varphi(t) = I_n$.

Since φ is a group homomorphism we have that $\varphi(s)\varphi(t) = \varphi(t)\varphi(s)$ for all $t, s \in \mathbb{R}$.

A commuting family of normal operators is simultaneously unitarily diagonalizable. Therefore, there exists S unitary and $\{v_i\}$ orthonormal basis such that for all $k = 1, \dots, n$ and for all $t \in \mathbb{R}$ we have:

$$\varphi(t)v_k = d_k(t)v_k, \quad S^*\varphi(t)S = \text{diag}(d_1(t), \dots, d_n(t)).$$

Note the following properties,

1. $d_k(t) = \langle \varphi(t)v_k, v_k \rangle$ is continuous due to continuity of the inner product.
2. $|d_k(t)| = 1$ since $\varphi(t)$ is norm preserving hence, $\|v_k\| = \|\varphi(t)v_k\| = \|d_k(t)v_k\| = |d_k(t)|\|v_k\|$.
3. $d_k(0) = 1$. since $\varphi(0) = I$.
4. $d_k(t+s)v_k = \varphi(t+s)v_k = \varphi(t)\varphi(s)v_k = \varphi(t)d_k(s)v_k = d_k(s)\varphi(t)v_k = d_k(s)d_k(t)v_k$.

Now by Lemma 2.5 we know there exist unique $\lambda_k \in \mathbb{R}$ such that $d_k(t) = e^{i\lambda_k t}$.

Let $D := \text{diag}(\lambda_1, \dots, \lambda_n)$, $H := SDS^*$. Note that $H^* = (SDS^*)^* = (S^*)^*D^*S^* = SDS^* = H$.

Lastly

$$e^{itH} = S\text{diag}(e^{i\lambda_1 t}, \dots, e^{i\lambda_n t})S^* = \varphi(t).$$

If $K \neq H$ satisfying the same conditions, then conjugate with S and compare diagonals. Uniqueness of scalars gives a contradiction. $\square$

2.2 Stone's theorem

The generalization to arbitrary Hilbert spaces requires the notion of the spectral theorem. It allows us to define exponentials for self adjoint operators $e^{itF} := \int_{\sigma(F)} e^{it\lambda} d\mu^F(\lambda)$.

Definition 2.7 (One parameter unitary group). *A one parameter unitary group is a family $U := \{U(t)\}_{t \in \mathbb{R}}$ of unitary operators such that $U(t+s) = U(t)U(s)$ and $U(-t) = U(t)^{-1}$.*

Additionally it is said to be strongly continuous if $t \mapsto U(t)v$ is continuous for every $v \in \mathcal{H}$.

Definition 2.8 (Infinitesimal generator). *For a strongly continuous one parameter unitary group U , the infinitesimal generator is the operator $Av = \lim_{t \rightarrow 0} \frac{1}{i} \frac{U(t)v - v}{t}$. The domain of the operator is defined for the set on which the limit is well defined in $\mathcal{H}$.*

Proposition 2.9 (Properties of $U(t)$). *Let A be self adjoint and define $U(t) = e^{itA}$ via the spectral calculus.*

1. $U(t)$ is a strongly continuous one-parameter unitary group.
2. The limit

$$\lim_{t \rightarrow 0} \frac{1}{i} \frac{U(t)v - v}{t}$$

exists if and only if $v \in \text{Dom}(A)$. When it exists, it equals Av .

3. $U(t) \text{Dom}(A) = \text{Dom}(A)$ and $AU(t) = U(t)A$.

Proof. Write $U(t) = \int e^{it\lambda} d\mu^A(\lambda)$.

(1) Note via PVM calculus we directly obtain $e^{it\lambda}e^{is\lambda} = e^{i(t+s)\lambda}$ and $|e^{it\lambda}| = 1$, the identities $U(t+s) = U(t)U(s)$ and $U(t)^* = U(-t)$ also follow from the PVM calculus (see Proposition B.20).

Strong continuity follows from

$$\|U(t)v - U(s)v\|^2 = \int |e^{it\lambda} - e^{is\lambda}|^2 d\mu_v^A(\lambda) \rightarrow 0$$

by dominated convergence using $|e^{it\lambda} - e^{is\lambda}| \leq 2$. This proves (1).

For (2), if $v \in \text{Dom}(A)$ then

$$\left\| \frac{U(t)v - v}{it} - Av \right\|^2 = \int \left| \frac{e^{it\lambda} - 1}{it} - \lambda \right|^2 d\mu_v^A(\lambda) \rightarrow 0,$$

by dominated convergence, since

$$\left| \frac{e^{it\lambda} - 1}{it} - \lambda \right| \leq \left| \frac{e^{it\lambda} - 1}{t} \right| + |\lambda| \leq 2|\lambda|,$$

so the squared integrand is dominated by $4\lambda^2$, which is μ_v^A -integrable because $v \in \text{Dom}(A)$.

Conversely, if the limit exists for v , it defines an operator $\tilde{A}$ extending A on $\text{Dom}(A)$ and $\tilde{A}$ is symmetric. By Proposition B.10 (self adjoint is maximal symmetric), $\tilde{A} = A$ and $v \in \text{Dom}(A)$.

For (3), fix t . For $v \in \text{Dom}(A)$,

$$\frac{U(h)U(t)v - U(t)v}{ih} = U(t) \frac{U(h)v - v}{ih} \rightarrow U(t)Av \quad (h \rightarrow 0).$$

Hence $U(t)v \in \text{Dom}(A)$ and $AU(t)v = U(t)Av$. $\square$

Lemma 2.10 (Dense domain). *Let $U : \mathbb{R} \rightarrow \mathcal{U}(\mathcal{H})$ be strongly continuous and define*

$$Av := \frac{1}{i} \lim_{t \rightarrow 0} \frac{U(t)v - v}{t}$$

on the set where the limit exists. Then $\text{Dom}(A)$ is dense in $\mathcal{H}$. More precisely, for $v \in \mathcal{H}$ and $\tau > 0$ set

$$v_\tau := \int_0^\tau U(s)v \, ds.$$

Then $v_\tau \in \text{Dom}(A)$, $\tau^{-1}v_\tau \rightarrow v$ as $\tau \rightarrow 0$, and

$$Av_\tau = \frac{1}{i}(U(\tau) - I)v.$$

Proof. Since $s \mapsto U(s)v$ is continuous, the integral exists in $\mathcal{H}$. Also,

$$\left\| \frac{1}{\tau}v_\tau - v \right\| \leq \frac{1}{\tau} \int_0^\tau \|U(s)v - v\| \, ds \xrightarrow{\tau \rightarrow 0} 0.$$

Next,

$$\begin{aligned} \frac{U(t)v_\tau - v_\tau}{t} &= \frac{1}{t} \int_0^\tau (U(t+s) - U(s))v \, ds \\ &= \frac{1}{t} \left(\int_\tau^{\tau+t} U(u)v \, du - \int_0^t U(u)v \, du \right) \longrightarrow (U(\tau) - I)v \end{aligned}$$

as $t \rightarrow 0$. This gives $v_\tau \in \text{Dom}(A)$ and the formula for Av_τ . Density follows from $\tau^{-1}v_\tau \rightarrow v$. $\square$

Lemma 2.11 (Symmetric). *The operator A is symmetric.*

Proof. For $v, w \in \text{Dom}(A)$,

$$\langle v, Aw \rangle = \lim_{t \rightarrow 0} \left\langle v, \frac{U(t)w - w}{it} \right\rangle = \lim_{t \rightarrow 0} \left\langle \frac{U(-t)v - v}{-it}, w \right\rangle = \langle Av, w \rangle.$$

□

Lemma 2.12 (Domain invariance and derivative). *If $v \in \text{Dom}(A)$ and $t \in \mathbb{R}$, then $U(t)v \in \text{Dom}(A)$ and*

$$AU(t)v = U(t)Av.$$

Also $t \mapsto U(t)v$ is differentiable and

$$\frac{d}{dt}U(t)v = iU(t)Av.$$

In particular,

$$U(t)v - v = i \int_0^t U(s)Av \, ds.$$

Proof. Fix $t \in \mathbb{R}$. Using $U(t+h) = U(t)U(h)$,

$$\frac{U(t+h)v - U(t)v}{h} = U(t) \frac{U(h)v - v}{h} \rightarrow iU(t)Av \quad (h \rightarrow 0).$$

So $U(t)v \in \text{Dom}(A)$ and the derivative formula holds. Dividing by i gives $AU(t)v = U(t)Av$. Integrating the derivative gives the integral identity. □

Lemma 2.13 (Closed). *The operator A is closed.*

Proof. Let $v_n \in \text{Dom}(A)$ with $v_n \rightarrow v$ and $Av_n \rightarrow w$ in $\mathcal{H}$. By Lemma 2.12,

$$U(t)v_n - v_n = i \int_0^t U(s)Av_n \, ds.$$

Let $n \rightarrow \infty$. Since $\|U(s)\| = 1$,

$$U(t)v - v = i \int_0^t U(s)w \, ds.$$

Then

$$\frac{1}{i} \frac{U(t)v - v}{t} = \frac{1}{t} \int_0^t U(s)w \, ds \rightarrow w \quad (t \rightarrow 0),$$

because $s \mapsto U(s)w$ is continuous at 0. So $v \in \text{Dom}(A)$ and $Av = w$. □

Lemma 2.14 (Kernel condition). *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\ker(A^* + z) = \{0\}, \quad \ker(A^* + z^*) = \{0\}.$$

Proof. Assume $w \in \ker(A^* + z)$, so $A^*w = -zw$. For $v \in \text{Dom}(A)$, by Lemma 2.12,

$$\frac{d}{dt} \langle w, U(t)v \rangle = \langle w, iAU(t)v \rangle = i \langle A^*w, U(t)v \rangle = -iz \langle w, U(t)v \rangle.$$

So $\langle w, U(t)v \rangle = e^{-izt} \langle w, v \rangle$. The left side is bounded in t (unitarity). If $\Im z \neq 0$, then $|e^{-izt}|$ is unbounded on $\mathbb{R}$. Hence $\langle w, v \rangle = 0$ for all $v \in \text{Dom}(A)$, and by Lemma 2.10, $w = 0$. The same argument works for z^* . $\square$

Lemma 2.15 (Uniqueness). *Let A be self-adjoint. Let $U(t)$ and $V(t)$ be strongly continuous one-parameter unitary groups. Assume A is the infinitesimal generator of both. Then $U(t) = V(t)$ for all $t \in \mathbb{R}$.*

Proof. Fix $v \in \text{Dom}(A)$ and set $r(t) := (U(t) - V(t))v$. Using Lemma 2.12 for both groups, r is differentiable and

$$r'(t) = iAr(t).$$

Hence

$$\frac{d}{dt} \|r(t)\|^2 = 2 \Re \langle r(t), r'(t) \rangle = 2 \Re \langle r(t), iAr(t) \rangle = 0,$$

since $\langle r, Ar \rangle \in \mathbb{R}$ for self-adjoint A . So $\|r(t)\|$ is constant, and $r(0) = 0$ gives $r(t) = 0$ for all t . By density of $\text{Dom}(A)$ and boundedness of $U(t), V(t)$, this extends to all of $\mathcal{H}$. $\square$

Theorem 2.16 (Stone). *Let $U(t)$ be a weakly continuous one-parameter unitary group on $\mathcal{H}$ and define*

$$Av := \frac{1}{i} \lim_{t \rightarrow 0} \frac{U(t)v - v}{t}$$

on the set where the limit exists. Then A is self adjoint and $U(t) = e^{itA}$ for all $t \in \mathbb{R}$.

Proof. By Lemma B.5, U is strongly continuous.

Lemma 2.10 gives density of $\text{Dom}(A)$. Lemma 2.11 gives symmetry of A . Lemma 2.14 and Lemma B.14 give that A is essentially self-adjoint. Lemma 2.13 shows A is closed, so A is self-adjoint.

Let $V(t) := e^{itA}$ defined by the spectral calculus. By Proposition 2.9, $V(t)$ is a strongly continuous unitary group with generator A . By Lemma 2.15, $U(t) = V(t)$ for all t . $\square$

3 Irreversible dynamics: Hille-Yosida theorem

We now turn our attention to time irreversible dynamics. As mentioned before these are modelled by semigroups. In particular we will no longer be studying group homomorphisms from $\mathbb{R}$ into some automorphism group of the hilbert space but rather semigroup morphisms into endomorphisms of a (complex) Banach space. We will be largely following the book by Engel and Nagel [EBN⁺99], and lecture notes by Loss [Los17].

The theorem which serves as the capstone for this section is the Hille-Yosida theorem. It provides us a condition for the existence of generator for a family of particularly ‘nice’ one parameter semigroups. In this sense it is best realized as the non-reversible analog to Stone’s theorem.

For completeness recall the definition of a Banach space.

Definition 3.1 (Banach space). *A complete normed complex vector space $(\mathcal{B}, \|\cdot\|)$ is a Banach space.*

3.1 C_0 semigroups and the generator

We will be discussing strongly continuous one parameter semigroups. These are commonly referred to as C_0 semigroups. We will stick to this convention.

Definition 3.2 (C_0 -semigroup). *A family $\{P_t\}_{t \geq 0}$ of bounded linear maps on a Banach space $\mathcal{B}$ is a C_0 -semigroup if*

1. $P_0 = I$
2. $P_{t+s} = P_t P_s$ for $t, s \geq 0$
3. $\lim_{t \rightarrow 0} \|P_t f - f\| = 0$ for all $f \in \mathcal{B}$.

It is a contraction semigroup if $\|P_t\| \leq 1$ for all $t \geq 0$.

The contraction hypothesis is natural for Hille–Yosida because:

1. It guarantees the Laplace integral $\int_0^\infty e^{-\lambda t} P_t f dt$ converges for $\operatorname{Re} \lambda > 0$ and yields the sharp bound $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\operatorname{Re} \lambda}$.

2. It is the norm-decreasing analogue of unitarity in Stone's theorem, giving a clean half-plane of resolvent boundedness.
3. It lets us approximate by genuine exponentials of bounded operators (Yosida approximants) while keeping $\|e^{tA_\lambda}\| \leq 1$ uniformly in λ and $t \geq 0$.

We now recall the notion of a generator.

Definition 3.3 (Generator). *Given a C_0 -semigroup $\{P_t\}$, define the generator A as such*

$$\text{Dom}(A) := \left\{ f \in \mathcal{B} : \lim_{h \downarrow 0} \frac{P_h f - f}{h} \text{ exists in } \mathcal{B} \right\}, \quad Af := \lim_{h \rightarrow 0} \frac{P_h f - f}{h}.$$

This is a similar notion of a generator as before.

1. If A is bounded, the exponential series $e^{tA} = \sum_{k \geq 0} \frac{t^k A^k}{k!}$ defines a C_0 -semigroup with $\frac{d}{dt} e^{tA} = A e^{tA}$ and $\frac{d}{dt} \Big|_{t=0} e^{tA} f = Af$. The definition above recovers A from any C_0 -semigroup by taking the strong derivative at 0.
2. For general (unbounded) A , the series may not make sense on all of $\mathcal{B}$, but the C_0 -semigroup still encodes e^{tA} the generator is the precise object that captures this.

Lemma 3.4 (Differentiability on $\text{Dom}(A)$). *If $f \in \text{Dom}(A)$, then for each $t \geq 0$ the limit*

$$\lim_{h \downarrow 0} \frac{P_{t+h} f - P_t f}{h}$$

exists in $\mathcal{B}$ and equals $P_t A f$. In particular $t \mapsto P_t f$ is differentiable on $[0, \infty)$ with $\frac{d}{dt} P_t f = P_t A f$ and $\frac{d}{dt} \Big|_{t=0} P_t f = A f$.

Proof. By the semigroup law,

$$\frac{P_{t+h} f - P_t f}{h} = \frac{P_t(P_h f - f)}{h} = P_t \left(\frac{P_h f - f}{h} \right) \longrightarrow P_t A f \quad (h \rightarrow 0),$$

since P_t is bounded and $f \in \text{Dom}(A)$. □

Proposition 3.5 (Density of the generator domain). *For a C_0 -semigroup $\{P_t\}$ on $\mathcal{B}$ with generator A , the set $\text{Dom}(A)$ is dense in $\mathcal{B}$.*

Proof. Fix $t > 0$ and define the averaging operator for $f \in \mathcal{B}$

$$V_t f := \frac{1}{t} \int_0^t P_s f \, ds,$$

where the integral is the Riemann integral of the continuous map $s \mapsto P_s f$ on $[0, t]$. Then V_t is bounded.

1. For $h > 0$,

$$\frac{P_h(V_t f) - V_t f}{h} = \frac{1}{t} \int_0^t \frac{P_{h+s} f - P_s f}{h} \, ds = \frac{1}{t} \left(\int_t^{t+h} P_u f \, du - \int_0^h P_u f \, du \right).$$

As $h \rightarrow 0$, $\frac{1}{h} \int_t^{t+h} P_u f \, du \rightarrow P_t f$ and $\frac{1}{h} \int_0^h P_u f \, du \rightarrow P_0 f = f$, so $V_t f \in \text{Dom}(A)$ and

$$A V_t f = \frac{P_t f - f}{t}. \quad (3.1)$$

2. By strong continuity at 0,

$$\|V_t f - f\| \leq \frac{1}{t} \int_0^t \|P_s f - f\| \, ds \xrightarrow{t \rightarrow 0} 0.$$

Hence $\bigcup_{t>0} \text{Ran}(V_t) \subset \text{Dom}(A)$ is dense in $\mathcal{B}$.

□

3.2 Laplace transform and the resolvent

In this subsection we will revisit the notion of a resolvent, and prove an equivalent formulation in terms of a Laplace transform which will be key in our proof of Hille-Yosida.

Definition 3.6 (Resolvent set and resolvent). *Let A be a densely defined operator on $\mathcal{B}$. The resolvent set is*

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective with bounded inverse}\}.$$

For $\lambda \in \rho(A)$ the resolvent operator at λ is

$$R_\lambda(A) := (\lambda I - A)^{-1}.$$

The map $\lambda \mapsto R_\lambda(A)$ on $\rho(A)$ is called the resolvent of A .

Let $\{P_t\}_{t \geq 0}$ be a contraction C_0 -semigroup on $\mathcal{B}$ with generator A .

With some abuse of notation we define the Laplace operator as such.

Definition 3.7 (Laplace operator $R_\lambda(A)$). *For $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$ define*

$$R_\lambda(A)f := \int_0^\infty e^{-\lambda t} P_t f dt \quad (f \in \mathcal{B}),$$

as an improper Riemann integral; the integrand $t \mapsto P_t f$ is continuous.

We will show that for appropriate λ this Laplace operator is indeed equivalent to the resolvent set.

Lemma 3.8 (Norm bound). *For $\operatorname{Re} \lambda > 0$,*

$$\|R_\lambda(A)\| \leq \frac{1}{\operatorname{Re} \lambda}.$$

Proof. For $f \in \mathcal{B}$,

$$\|R_\lambda(A)f\| \leq \int_0^\infty e^{-(\operatorname{Re} \lambda)t} \|P_t f\| dt \leq \|f\| \int_0^\infty e^{-(\operatorname{Re} \lambda)t} dt = \frac{1}{\operatorname{Re} \lambda} \|f\|.$$

□

Lemma 3.9 ($R_\lambda(A)$ maps into $\operatorname{Dom}(A)$ and solves the resolvent equation). *For $\operatorname{Re} \lambda > 0$ and $f \in \mathcal{B}$, $R_\lambda(A)f \in \operatorname{Dom}(A)$ and $(\lambda I - A)R_\lambda(A)f = f$.*

Proof. For $\varepsilon > 0$ set $F_\varepsilon := \int_\varepsilon^\infty e^{-\lambda t} P_t dt$. Then

$$\frac{P_\varepsilon - I}{\varepsilon} R_\lambda(A) = \frac{e^{\lambda\varepsilon} - 1}{\varepsilon} F_\varepsilon - \frac{1}{\varepsilon} \int_0^\varepsilon e^{-\lambda t} P_t dt.$$

Apply both sides to f . Since $\frac{e^{\lambda\varepsilon} - 1}{\varepsilon} \rightarrow \lambda$, $F_\varepsilon \rightarrow R_\lambda(A)$ in operator norm (by dominated convergence and Lemma 3.8), and

$$\frac{1}{\varepsilon} \int_0^\varepsilon e^{-\lambda t} P_t f dt \rightarrow f \quad (\varepsilon \rightarrow 0)$$

by continuity of $t \mapsto P_t f$, we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{P_\varepsilon R_\lambda(A)f - R_\lambda(A)f}{\varepsilon} = \lambda R_\lambda(A)f - f.$$

By the definition of the generator, this shows $R_\lambda(A)f \in \operatorname{Dom}(A)$ and $AR_\lambda(A)f = \lambda R_\lambda(A)f - f$, i.e. $(\lambda I - A)R_\lambda(A)f = f$. □

Lemma 3.10 (Right inverse on $\text{Dom}(A)$). *For $\text{Re } \lambda > 0$ and $f \in \text{Dom}(A)$,*

$$R_\lambda(A)(\lambda I - A)f = f.$$

Proof. For $f \in \text{Dom}(A)$, $t \mapsto P_t f$ is differentiable with $\frac{d}{dt} P_t f = A P_t f = P_t A f$. Compute

$$\frac{d}{dt} (e^{-\lambda t} P_t f) = e^{-\lambda t} (A P_t f - \lambda P_t f) = -e^{-\lambda t} (\lambda I - A) P_t f.$$

Integrating on $[0, T]$,

$$\int_0^T e^{-\lambda t} (\lambda I - A) P_t f dt = - \left[e^{-\lambda t} P_t f \right]_0^T = f - e^{-\lambda T} P_T f.$$

Let $T \rightarrow \infty$. Since $\text{Re } \lambda > 0$, $\|e^{-\lambda T} P_T\| \leq e^{-(\text{Re } \lambda)T} \rightarrow 0$, hence

$$R_\lambda(A)(\lambda I - A)f = \int_0^\infty e^{-\lambda t} (\lambda I - A) P_t f dt = f.$$

□

Corollary 3.11 (Right half-plane in the resolvent). *For $\text{Re } \lambda > 0$, $R_\lambda(A) = (\lambda I - A)^{-1}$. In particular, $\{\lambda : \text{Re } \lambda > 0\} \subset \rho(A)$.*

Proof. By Lemma 3.9 we have $(\lambda I - A)R_\lambda(A) = I$ on $\mathcal{B}$. By Lemma 3.10 we have $R_\lambda(A)(\lambda I - A) = I$ on $\text{Dom}(A)$. Hence $R_\lambda(A)$ is the bounded two-sided inverse of $\lambda I - A$. □

3.3 Yosida approximants

Our goal is to find necessary and sufficient conditions for a closed operator to be a generator for C_0 semigroup on some Banach space. Our discussion on the Laplace operator has already gotten us halfway there.

The difficult half of the proof of Hille-Yosida involves showing that subject to some conditions on the resolvent set we can find a generator A . We will find this A using a sequence of approximant operators which converge to A . These sequences are what are known as Yosida approximants.

Assume A is closed and densely defined on $\mathcal{B}$, $(0, \infty) \subset \rho(A)$, and

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \text{ for } \lambda > 0.$$

Lemma 3.12. For $f \in \mathcal{B}$, $\lambda(\lambda I - A)^{-1}f \rightarrow f$ as $\lambda \rightarrow \infty$.

Proof. First $f \in \text{Dom}(A)$:

$$\lambda(\lambda I - A)^{-1}f - f = (\lambda I - A)^{-1}Af,$$

hence

$$\|\lambda(\lambda I - A)^{-1}f - f\| \leq \frac{1}{\lambda} \|Af\| \xrightarrow{\lambda \rightarrow \infty} 0.$$

For arbitrary f , given $\varepsilon > 0$ choose $g \in \text{Dom}(A)$ with $\|f - g\| < \varepsilon$. Then

$$\begin{aligned} \|\lambda(\lambda I - A)^{-1}f - f\| &\leq \|\lambda(\lambda I - A)^{-1}(f - g) - (f - g)\| \\ &\quad + \|\lambda(\lambda I - A)^{-1}g - g\| \\ &\leq 2\varepsilon + o(1). \end{aligned}$$

Let $\lambda \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ and we are done. $\square$

Lemma 3.13. The operator $\lambda(\lambda I - A)^{-1}A$ extends to a bounded operator on $\mathcal{B}$ and, for $f \in \text{Dom}(A)$,

$$\|\lambda(\lambda I - A)^{-1}Af - Af\| \xrightarrow{\lambda \rightarrow \infty} 0.$$

Proof. Use

$$\lambda(\lambda I - A)^{-1}A = \lambda(\lambda I - A)^{-1}(A - \lambda I) + \lambda^2(\lambda I - A)^{-1} = \lambda^2(\lambda I - A)^{-1} - \lambda I.$$

Thus for $f \in \text{Dom}(A)$,

$$\|\lambda(\lambda I - A)^{-1}Af\| = \|\lambda^2(\lambda I - A)^{-1}f - \lambda f\| \leq \lambda \|f\| + \lambda \|f\| = 2\lambda \|f\|,$$

and density of $\text{Dom}(A)$ yields boundedness. The convergence on $\text{Dom}(A)$ follows from Lemma 3.12. $\square$

Definition 3.14 (Yosida approximants). For $\lambda > 0$ define the bounded operator

$$A_\lambda := \lambda(\lambda I - A)^{-1}A.$$

Lemma 3.15. For each $\lambda > 0$ and $t \geq 0$ the series

$$e^{A_\lambda t} := \sum_{k=0}^{\infty} \frac{(A_\lambda t)^k}{k!}$$

converges in operator norm and $(e^{A_\lambda t})_{t \geq 0}$ is a contraction semigroup.

Proof. Norm convergence and the semigroup law are standard for bounded generators. For the contraction bound,

$$\begin{aligned}\|e^{A_\lambda t}\| &= \left\| e^{t(\lambda^2(\lambda I - A)^{-1} - \lambda I)} \right\| \leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \|\lambda^2(\lambda I - A)^{-1}\|^k \\ &\leq e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} = 1,\end{aligned}$$

since $\|\lambda^2(\lambda I - A)^{-1}\| \leq \lambda$. □

3.4 Hille–Yosida theorem

We now have all the data required to state and prove the Hille-Yosida theorem.

Lemma 3.16 (Integral identity for $e^{A_\lambda t} - e^{A_\mu t}$). *Fix $\lambda, \mu > 0$. Then $A_\lambda A_\mu = A_\mu A_\lambda$. For $t \geq 0$,*

$$e^{A_\lambda t} - e^{A_\mu t} = \int_0^t e^{A_\mu(t-s)} (A_\lambda - A_\mu) e^{A_\lambda s} ds.$$

In particular, for $f \in \mathcal{B}$,

$$\| [e^{A_\lambda t} - e^{A_\mu t}] f \| \leq t \| (A_\lambda - A_\mu) f \|. \quad (*)$$

Proof. $(\lambda I - A)$ and $(\mu I - A)$ commute. Their inverses commute. So A_λ and A_μ commute.

Set $F(s) := e^{A_\mu(t-s)} e^{A_\lambda s}$. Then

$$F'(s) = e^{A_\mu(t-s)} (A_\lambda - A_\mu) e^{A_\lambda s}.$$

Integrate F' from 0 to t . This gives the identity.

For the estimate use $\|e^{A_\lambda s}\| \leq 1$ and $\|e^{A_\mu(t-s)}\| \leq 1$ (Lemma 3.15). □

Lemma 3.17 (Limit on $\text{Dom}(A)$). *Fix $T > 0$ and $f \in \text{Dom}(A)$. Then $e^{A_\lambda t} f$ is Cauchy uniformly for $t \in [0, T]$. Hence for each $t \geq 0$ the limit*

$$P_t f := \lim_{\lambda \rightarrow \infty} e^{A_\lambda t} f$$

exists.

Proof. By Lemma 3.13 we have $A_\lambda f \rightarrow Af$. So $\|(A_\lambda - A_\mu) f\| \rightarrow 0$ as $\lambda, \mu \rightarrow \infty$. Now use $(*)$ with $t \leq T$. □

Lemma 3.18 (Extend to all of $\mathcal{B}$). *For each $t \geq 0$ the limit*

$$P_t f := \lim_{\lambda \rightarrow \infty} e^{A_\lambda t} f$$

exists for all $f \in \mathcal{B}$. Also $\|P_t\| \leq 1$.

Proof. First take $f \in \text{Dom}(A)$. This is Lemma 3.17.

Now take general $f \in \mathcal{B}$. Choose $g \in \text{Dom}(A)$ with $\|f - g\|$ small. For every λ ,

$$\|e^{A_\lambda t} f - e^{A_\lambda t} g\| \leq \|f - g\|$$

since $\|e^{A_\lambda t}\| \leq 1$. This lets us define $P_t f$ by density. The contraction bound passes to the limit. $\square$

Lemma 3.19 (Semigroup law). *$P_0 = I$ and $P_{t+s} = P_t P_s$ for $t, s \geq 0$.*

Proof. $P_0 = I$ is clear.

Fix $f \in \mathcal{B}$. Then

$$P_{t+s} f = \lim_{\lambda \rightarrow \infty} e^{A_\lambda(t+s)} f = \lim_{\lambda \rightarrow \infty} e^{A_\lambda t} e^{A_\lambda s} f.$$

Write

$$e^{A_\lambda t} e^{A_\lambda s} f - e^{A_\lambda t} P_s f = e^{A_\lambda t} (e^{A_\lambda s} - P_s) f.$$

The norm goes to 0 since $\|e^{A_\lambda t}\| \leq 1$ and $e^{A_\lambda s} f \rightarrow P_s f$. So

$$P_{t+s} f = \lim_{\lambda \rightarrow \infty} e^{A_\lambda t} P_s f = P_t(P_s f).$$

$\square$

Lemma 3.20 (Strong continuity at 0). *For every $f \in \mathcal{B}$ we have $\|P_t f - f\| \rightarrow 0$ as $t \downarrow 0$.*

Proof. Fix $T > 0$ and $f \in \text{Dom}(A)$. By (*), for $0 \leq t \leq T$,

$$\| [e^{A_\lambda t} - e^{A_\mu t}] f \| \leq T \| (A_\lambda - A_\mu) f \|.$$

The right side goes to 0 since $A_\lambda f \rightarrow A f$. So $e^{A_\lambda t} f$ is Cauchy uniformly on $[0, T]$. Thus $P_t f$ is a uniform limit of continuous maps. So $t \mapsto P_t f$ is continuous on $[0, T]$. In particular, $\|P_t f - f\| \rightarrow 0$ as $t \downarrow 0$.

Now take general $f \in \mathcal{B}$. Choose $g \in \text{Dom}(A)$ close to f . Use $\|P_t\| \leq 1$:

$$\|P_t f - f\| \leq 2\|f - g\| + \|P_t g - g\|.$$

Let $t \downarrow 0$ and then $\|f - g\| \downarrow 0$. $\square$

Lemma 3.21 (Resolvent formula for A_μ). Fix $\lambda > 0$ and $\mu > 0$. Set $\alpha := \frac{\lambda\mu}{\lambda + \mu}$. Then

$$(\lambda I - A_\mu)^{-1} = \frac{1}{\lambda + \mu} I + \left(\frac{\mu}{\lambda + \mu} \right)^2 (\alpha I - A)^{-1}.$$

In particular, as $\mu \rightarrow \infty$,

$$(\lambda I - A_\mu)^{-1} \rightarrow (\lambda I - A)^{-1}$$

in operator norm.

Proof. Use $A_\mu = \mu^2(\mu I - A)^{-1} - \mu I$. Compute

$$(\lambda I - A_\mu)(\mu I - A) = (\lambda + \mu)(\alpha I - A).$$

Invert this identity. This gives the displayed formula.

For the limit: $\alpha \rightarrow \lambda$ and $\left(\frac{\mu}{\lambda + \mu} \right)^2 \rightarrow 1$. Also

$$(\alpha I - A)^{-1} - (\lambda I - A)^{-1} = (\lambda - \alpha)(\alpha I - A)^{-1}(\lambda I - A)^{-1}.$$

Use $\|(\beta I - A)^{-1}\| \leq 1/\beta$ for $\beta > 0$. So the norm goes to 0 as $\mu \rightarrow \infty$. $\square$

Lemma 3.22 (The generator is A). Let $\{P_t\}_{t \geq 0}$ be the semigroup from Lemmas 3.18–3.20. Let G be its generator. Then $G = A$.

Proof. Fix $\lambda > 0$ and set

$$R_\lambda f := \int_0^\infty e^{-\lambda t} P_t f \, dt.$$

By Corollary 3.11 applied to $\{P_t\}$,

$$R_\lambda = (\lambda I - G)^{-1}.$$

For each $\mu > 0$, Corollary 3.11 applied to $(e^{A_\mu t})_{t \geq 0}$ gives

$$\int_0^\infty e^{-\lambda t} e^{A_\mu t} f \, dt = (\lambda I - A_\mu)^{-1} f.$$

Now $e^{A_\mu t} f \rightarrow P_t f$ for each t and $\|e^{A_\mu t} f\| \leq \|f\|$. So dominated convergence gives

$$R_\lambda f = \lim_{\mu \rightarrow \infty} (\lambda I - A_\mu)^{-1} f.$$

By Lemma 3.21, this equals $(\lambda I - A)^{-1} f$. So $(\lambda I - G)^{-1} = (\lambda I - A)^{-1}$ for all $\lambda > 0$. Hence $G = A$. $\square$

Theorem 3.23 (Hille–Yosida). *A closed densely defined operator A is the generator of a contraction C_0 –semigroup on $\mathcal{B}$ if and only if*

$$(0, \infty) \subset \rho(A) \quad \text{and} \quad \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \quad \text{for all } \lambda > 0.$$

Proof. The forward direction is given by the Laplace–resolvent lemmas.

For the converse, define the Yosida approximants A_λ (Definition 3.14). Set $P_t^{(\lambda)} := e^{A_\lambda t}$ (Lemma 3.15). Define $P_t f := \lim_{\lambda \rightarrow \infty} P_t^{(\lambda)} f$.

Lemmas 3.18, 3.19, and 3.20 show that $\{P_t\}$ is a contraction C_0 –semigroup.

Let G be its generator. Lemma 3.22 gives $G = A$. $\square$

4 Some physical examples

In this section we sketch a few models where the abstract theory above appears in a natural way.

4.1 Schrödinger dynamics via Stone’s theorem

For this section we refer to [Rem].

Let $\mathcal{H}$ be a complex Hilbert space. A time evolution is a strongly continuous one-parameter unitary group $(U(t))_{t \in \mathbb{R}}$ on $\mathcal{H}$.

By Stone’s theorem we know there is a unique self-adjoint operator H such that

$$U(t) = e^{-itH} \quad (t \in \mathbb{R}).$$

In quantum mechanics the operator H is called the Hamiltonian.

Proposition 4.1 (Hilbert-space Schrödinger equation). *Fix $\psi_0 \in \mathcal{H}$ and define $\psi(t) := U(t)\psi_0$. Then $\|\psi(t)\| = \|\psi_0\|$ for all t . If moreover $\psi_0 \in \text{Dom}(H)$, then $t \mapsto \psi(t)$ is differentiable in $\mathcal{H}$ and*

$$i \frac{d}{dt} \psi(t) = H\psi(t), \quad \psi(0) = \psi_0.$$

Proof. Norm preservation is immediate from unitarity.

Assume $\psi_0 \in \text{Dom}(H)$. By the definition of the generator,

$$H\psi_0 = \frac{1}{i} \lim_{h \rightarrow 0} \frac{U(h)\psi_0 - \psi_0}{h}.$$

Using the group law,

$$\frac{\psi(t+h) - \psi(t)}{h} = \frac{U(t+h)\psi_0 - U(t)\psi_0}{h} = U(t) \frac{U(h)\psi_0 - \psi_0}{h}.$$

Taking $h \rightarrow 0$ gives $\psi'(t) = -iU(t)H\psi_0$. Since $U(t) = e^{-itH}$ commutes with H on $\text{Dom}(H)$, this equals $-iH\psi(t)$. $\square$

4.2 The heat equation on $\mathbb{R}^n$

We work out an example as seen in [Los18]. The heat equation is

$$\partial_t u(t, x) = \Delta u(t, x), \quad u(0, x) = f(x), \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

We solve it in the space $X = L^2(\mathbb{R}^n)$.

Let $\widehat{f}$ denote the L^2 Fourier transform. Define an operator A by

$$D(A) = \{f \in L^2(\mathbb{R}^n) : |\xi|^2 \widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}, \quad \widehat{Af}(\xi) = -|\xi|^2 \widehat{f}(\xi).$$

So A is the L^2 Laplacian.

Lemma 4.2. *The operator A is densely defined and closed.*

Proof. Density: $\mathcal{S}(\mathbb{R}^n) \subset D(A)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in L^2 .

Closedness: take $f_k \in D(A)$ with $f_k \rightarrow f$ in L^2 and $Af_k \rightarrow g$ in L^2 . Plancherel gives $\widehat{f_k} \rightarrow \widehat{f}$ and $\widehat{Af_k} \rightarrow \widehat{g}$ in L^2 . But $\widehat{Af_k} = -|\xi|^2 \widehat{f_k}$. So $|\xi|^2 \widehat{f_k} \rightarrow -\widehat{g}$ in L^2 . Hence $|\xi|^2 \widehat{f} \in L^2$ and $\widehat{Af} = \widehat{g}$. So $f \in D(A)$ and $Af = g$. $\square$

Lemma 4.3. *For every $\lambda > 0$ define $R(\lambda) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by*

$$(\widehat{R(\lambda)f})(\xi) = \frac{1}{\lambda + |\xi|^2} \widehat{f}(\xi).$$

Then $R(\lambda) = (\lambda I - A)^{-1}$ and

$$\|R(\lambda)\|_{L^2 \rightarrow L^2} \leq \frac{1}{\lambda}.$$

Proof. We compute

$$(\lambda I - A)R(\lambda)f(\xi) = (\lambda + |\xi|^2) \frac{1}{\lambda + |\xi|^2} \widehat{f}(\xi) = \widehat{f}(\xi).$$

So $(\lambda I - A)R(\lambda) = I$. Thus $\lambda \in \rho(A)$ and $R(\lambda) = (\lambda I - A)^{-1}$.

For the bound use Plancherel and $\left| \frac{1}{\lambda + |\xi|^2} \right| \leq \frac{1}{\lambda}$:

$$\|R(\lambda)f\|_2^2 = \int_{\mathbb{R}^n} \frac{1}{(\lambda + |\xi|^2)^2} |\widehat{f}(\xi)|^2 d\xi \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi = \frac{1}{\lambda^2} \|f\|_2^2.$$

□

Proposition 4.4. *The operator A generates a contraction C_0 semigroup $(P_t)_{t \geq 0}$ on $L^2(\mathbb{R}^n)$.*

Proof. By Lemma 4.2, A is closed and densely defined. By Lemma 4.3, $(0, \infty) \subset \rho(A)$ and $\|(\lambda I - A)^{-1}\| \leq 1/\lambda$ for all $\lambda > 0$. So the Hille–Yosida theorem (in the contraction form) applies.

□

Proposition 4.5. *For $t \geq 0$ and $f \in L^2(\mathbb{R}^n)$ we have*

$$(\widehat{P_t f})(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi).$$

Proof. Define S_t on L^2 by $(\widehat{S_t f})(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi)$. Then $\|S_t\| \leq 1$ and $S_{t+s} = S_t S_s$. So (S_t) is a contraction semigroup.

If $f \in D(A)$ then $|\xi|^2 \widehat{f} \in L^2$ and

$$\frac{e^{-h|\xi|^2} - 1}{h} \widehat{f}(\xi) \rightarrow -|\xi|^2 \widehat{f}(\xi) \quad \text{in } L^2 \text{ as } h \downarrow 0,$$

by dominated convergence using $\left| \frac{e^{-h|\xi|^2} - 1}{h} \right| \leq |\xi|^2$. So the generator of (S_t) is A . By uniqueness of the C_0 semigroup with generator A , we get $S_t = P_t$. □

Corollary 4.6. *If $f \in D(A)$ and $u(t) = P_t f$, then u is differentiable in L^2 and $u'(t) = Au(t)$. So u solves $\partial_t u = \Delta u$ in the L^2 sense.*

Proof. This is the standard semigroup fact: for $f \in D(A)$ we have $\frac{d}{dt} P_t f = A P_t f$. □

4.3 Markov semigroups

See [Sha07, Don03] for more details regarding the use of semigroups in probability theory. Here we just list a natural probabilistic occurrence of C_0 contraction semigroups. Although we do not need to use Hille–Yosida explicitly.

We begin by recalling some basic notions from probability theory.

Definition 4.7 (Poisson distribution). *Fix $\lambda > 0$. A random variable K has the $\text{Poisson}(\lambda t)$ distribution if*

$$\mathbb{P}(K = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad (k \in \mathbb{N}_0).$$

Definition 4.8 (Poisson process). *A Poisson process $\{N_t\}_{t \geq 0}$ of rate λ is an $\mathbb{N}_0$ -valued process with $N_0 = 0$, stationary independent increments, and*

$$N_t - N_s \sim \text{Poisson}(\lambda(t - s)) \quad (0 \leq s \leq t).$$

N_t counts how many events happened up to time t when events occur ‘independently at constant rate λ ’.

Define the following state space and test functions.

$$E = \mathbb{N}_0, \quad X = c_0(E) = \{f : E \rightarrow \mathbb{R} : f(n) \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

We may interpret $f \in X$ as a bounded ‘observable’ of the state n .

Definition 4.9 (Markov semigroup). *For $t \geq 0$ define $P_t : X \rightarrow X$ by*

$$(P_t f)(n) := \mathbb{E}[f(n + N_t)] = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} f(n + k).$$

Lemma 4.10. $\{P_t\}_{t \geq 0}$ is a contraction C_0 -semigroup on $(X, \|\cdot\|_{\infty})$.

Proof. Fix $t \geq 0$ and $f \in X$.

- First, $P_t f \in X$: since $f(n) \rightarrow 0$, given $\varepsilon > 0$ choose N with $|f(m)| < \varepsilon$ for all $m \geq N$. Then for $n \geq N$,

$$|(P_t f)(n)| \leq \sum_{k \geq 0} e^{-\lambda t} \frac{(\lambda t)^k}{k!} |f(n + k)| \leq \varepsilon \sum_{k \geq 0} e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \varepsilon,$$

so $(P_t f)(n) \rightarrow 0$.

- Contraction: for each n the coefficients are nonnegative and sum to 1, hence

$$|(P_t f)(n)| \leq \sum_{k \geq 0} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \|f\|_{\infty} = \|f\|_{\infty},$$

so $\|P_t f\|_{\infty} \leq \|f\|_{\infty}$. Also $P_0 = I$ since $N_0 = 0$.

- Semigroup law: for $s, t \geq 0$, write $N_{t+s} = N_t + M$ with $M := N_{t+s} - N_t$ independent of N_t and $M \stackrel{d}{=} N_s$. Then for each n ,

$$\begin{aligned}(P_{t+s}f)(n) &= \mathbb{E}[f(n + N_t + M)] = \mathbb{E}[\mathbb{E}[f(n + N_t + M) \mid N_t]] \\ &= \mathbb{E}[(P_sf)(n + N_t)] = (P_t(P_sf))(n).\end{aligned}$$

- Strong continuity at 0:

$$\begin{aligned}\|P_tf - f\|_\infty &\leq |1 - e^{-\lambda t}| \|f\|_\infty + \left(\sum_{k \geq 1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \right) \|f\|_\infty \\ &= 2(1 - e^{-\lambda t}) \|f\|_\infty \rightarrow 0.\end{aligned}$$

For general t_0 , use $P_t - P_{t_0} = P_{t_0}(P_{t-t_0} - I)$ and $\|P_{t_0}\| \leq 1$.

□

Proposition 4.11 (Generator). *The generator A of $\{P_t\}$ is given on all of X by*

$$(Af)(n) = \lambda(f(n+1) - f(n)).$$

Equivalently, if $(Sf)(n) := f(n+1)$ is the shift, then $A = \lambda(S - I)$.

Proof. Let $S : X \rightarrow X$ be the shift $(Sf)(n) = f(n+1)$. Then S is bounded on X and $\|S\| \leq 1$. Using the series for the exponential,

$$P_t = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} S^k = e^{-\lambda t} e^{\lambda t S} = e^{t\lambda(S-I)}.$$

Since $B := \lambda(S - I)$ is bounded, $\{e^{tB}\}$ is a uniformly continuous C_0 -semigroup with generator B . Therefore the generator of $\{P_t\}$ is $A = B = \lambda(S - I)$, i.e. $(Af)(n) = \lambda(f(n+1) - f(n))$. □

In this example the generator $A = \lambda(S - I)$ is bounded, so we do not really need Hille–Yosida to find it. The semigroup can be built directly from the explicit formula above. Hille–Yosida matters when the natural generator is unbounded.

5 Neretin's semigroup of annuli containing $\text{Diff}^+(S^1)$

This section follows Neretin's construction [Ner87]. Neretin is motivated by the following fact. If G is a real semisimple Lie group and $C \subset \mathfrak{g}$ is an invariant convex cone, then $G \exp(iC)$ is an open semigroup inside the complexified group $G^{\mathbb{C}}$.

For $G = \text{Diff}^+(S^1)$ there is no natural complex Lie group $\text{Diff}^{\mathbb{C}}$. In spite of this we can construct a complex semigroup which gives a concrete semigroup that contains $\text{Diff}^+(S^1)$ as the boundary case.

5.1 Normal form $(p, R(it), q)$

We realise the circle as $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $G = \text{Diff}^+(S^1)$ denote the group of orientation-preserving *real-analytic* diffeomorphisms of S^1 .

Example 5.1. *Some elements of $\text{Diff}^+(S^1)$*

- *Rotations:* $\phi(\theta) = \theta + \alpha$.
- *Flows:* if $v : S^1 \rightarrow \mathbb{R}$ is real-analytic, then its time- t flow gives a real-analytic diffeomorphism of S^1 .
- *Disk automorphisms:*

$$m_{a,\alpha}(z) = e^{i\alpha} \frac{z - a}{1 - \bar{a}z}, \quad |a| < 1.$$

Restricted to $|z| = 1$, this gives an element of $\text{Diff}^+(S^1)$.

Definition 5.2 (Radial contraction $R(it)$). *For $t > 0$ let $R(it) : \mathbb{C} \rightarrow \mathbb{C}$ be the map*

$$R(it)(z) = e^{-t}z.$$

We can define elements of the semigroup of annuli as a triplet

$$(p, R(it), q).$$

A doubly connected region (an annulus) specified by $R(it)$ together with its two boundary parametrizations into a triple $(p, R(it), q)$, where $t > 0$ and $p, q \in \text{Diff}^+(S^1)$, with the normalization $p(1) = 1$.

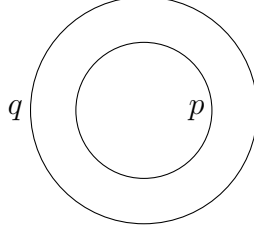


Figure 1: An element of the semigroup in the form of the triplet $(p, R(it), q)$

5.2 Canonical decomposition

We work with a class of analytic boundary maps.

Definition 5.3 (The set M). *Let M be the set of analytic maps $\rho : S^1 \rightarrow \{z \in \mathbb{C} : |z| < 1\}$ such that*

$$\rho'(z) \neq 0 \text{ for } z \in S^1, \quad \rho(S^1) \text{ is a positively oriented Jordan curve.}$$

Definition 5.4 (Canonical decomposition). *Let $\rho \in M$. A canonical decomposition of ρ is a triple $(p, R(it), q)$ with $t > 0$ and $p, q \in \text{Diff}^+(S^1)$ such that:*

- $p(1) = 1$
- p^{-1} maps conformally the doubly connected region between the outer circle S^1 and the inner contour $\rho(S^1)$ onto the round annulus

$$A_t = \{e^{-t} \leq |z| \leq 1\}$$

- on the boundary S^1 we have

$$\rho = p \circ R(it) \circ q.$$

If $\rho_1 = (p_1, R(it_1), q_1)$ and $\rho_2 = (p_2, R(it_2), q_2)$, you want to glue output of ρ_2 to input of ρ_1 .

The middle boundary map becomes $q_1 p_2$. Problem: $R(it_1) \circ (q_1 p_2) \circ R(it_2)$ is not in round form. So we must re-round using canonical decomposition, repeatedly.

Example 5.5 (Gluing can produce a non-round annulus). Let $m_a(z) = \frac{z-a}{1-\bar{a}z}$ with $0 < |a| < 1$. Fix $t_1, t_2 > 0$.

$$\gamma(\theta) = e^{-t_1} m_a(e^{-t_2} e^{i\theta}).$$

Then γ is a circle not centered at 0 if $a \neq 0$. So the glued domain is not a round annulus A_s , hence we must re-round via Π . The semigroup model

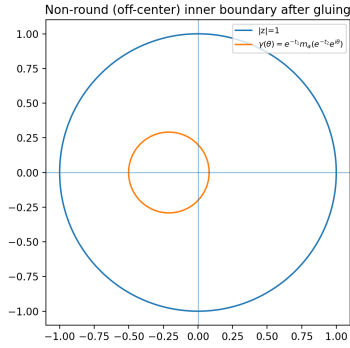


Figure 2: Here $a = 0.4, t_1 = 0.4, t_2 = 0.7$

only stores “round + boundary data”, so after gluing we must canonically uniformize the doubly connected domain back to a valid form and record the new boundary diffeomorphisms.

It is worth spelling out what the parameter t really is. Let $\Omega(\rho)$ be the doubly connected region between $|z| = 1$ and $\rho(S^1)$.

Note that for any doubly connected domains, there is a unique number $r \in (0, 1)$ such that $\Omega(\rho)$ is conformally equivalent to $\{r \leq |z| \leq 1\}$. See Chapter 6 Section 5 Th. 10 in [Ahl] specialized at $n = 2$.

We may encode this by $r = e^{-t}$, i.e. $t = -\log r$. Equivalently: t is the ‘modulus parameter’ of the annulus in the normal form.

Lemma 5.6 (Uniqueness of t and the normalization $p(1) = 1$). Assume $\rho \in M$ admits a canonical decomposition. Then t is uniquely determined. Moreover, for fixed t , the map p is determined up to a rotation of the round annulus, and the condition $p(1) = 1$ removes this rotational ambiguity.

Proof. Uniqueness of t is exactly uniqueness of the modulus parameter of a doubly connected domain. For the model annulus $A_t = \{e^{-t} \leq |z| \leq 1\}$, any

conformal automorphism that preserves both boundary circles is a rotation. Hence p is determined up to a rotation, and $p(1) = 1$ fixes it. $\square$

5.3 Univalent continuation

To multiply two triples we need to evaluate circle diffeomorphisms on circles of radii different from 1, so we must know how they extend off S^1 .

Definition 5.7 (Univalent continuation to an annulus). *Let $r \in \text{Diff}^+(S^1)$. We say r admits a univalent continuation to the annulus $e^{-\alpha} \leq |z| \leq e^\alpha$ if r extends to a holomorphic injective map on some neighbourhood of that annulus.*

Neretin's first lemma says that canonical decomposition is stable after inserting a small collar. In the proof one needs to extend conformal maps slightly past analytic boundary curves. This is done by what Neretin calls the symmetry principle. This most likely refers to Schwarz reflection principle. A conformal map on one side of an analytic boundary arc extends holomorphically across that arc after reflecting.

Lemma 5.8 (Neretin's Lemma 1). *Suppose $r \in \text{Diff}^+(S^1)$ admits a univalent continuation to the annulus $e^{-\varepsilon} \leq |z| \leq e^\varepsilon$, and assume its image lies in the disc $|z| < e^t$ for some $t > 0$. For some fixed D with $0 < D < \varepsilon$, and*

$$\rho = R(it) \circ r \circ R(iD).$$

Let $(p, R(is), q)$ be the canonical decomposition of ρ . Then:

- *q admits a univalent continuation to the larger annulus*

$$U := \{e^{-\varepsilon-D} \leq |z| \leq e^{\varepsilon+D}\}.$$

- *The image $q(U)$ is contained in the disc $|z| < e^s$.*
- *p admits a univalent continuation such that the identity $pR(is)q = R(it)rR(iD)$ holds as an identity of holomorphic maps on an annulus neighbourhood of S^1 .*

Proof. Because r extends univalently to $\{e^{-\varepsilon} \leq |z| \leq e^\varepsilon\}$, the composite $\rho = R(it) \circ r \circ R(iD)$ extends univalently to a neighbourhood of S^1 . Moreover,

the scaling $R(it)$ forces $\rho(S^1) \subset \{|z| < 1\}$, so $\rho(S^1)$ is an analytic Jordan curve.

Let Ω be the doubly connected region between the outer circle S^1 and the inner curve $\rho(S^1)$. By definition of canonical decomposition, p^{-1} is a conformal map $\Omega \rightarrow A_s$, and $p(1) = 1$ fixes the normalisation.

Now use the symmetry principle (Schwarz reflection) across analytic boundary curves: a conformal map between domains with analytic boundary extends holomorphically to a small collar past the boundary. Apply this to p^{-1} across both boundary components of Ω . After inverting, p also extends to a collar neighbourhood.

Finally, on S^1 we have $\rho = pR(is)q$, hence $q = R(-is)p^{-1}\rho$ on S^1 . All three factors extend holomorphically to a collar, so q extends there as well. This gives (a), (b), and the extended identity in (c). $\square$

5.4 The exhaustion procedure Π , semigroup product

The problem with naïve multiplication is that

$$R(it_1) \circ (q_1 p_2) \circ R(it_2)$$

is not of the form $R(is)$, so we must re-factor it back into normal form. Neretin does this by slicing t_2 into many small pieces and repeatedly applying canonical decomposition.

Definition 5.9 (Exhaustion procedure Π). *Fix $t > 0$ and $\tau > 0$. Let $r \in \text{Diff}^+(S^1)$. Assume r admits a univalent continuation to an annulus $e^{-\alpha} \leq |z| \leq e^\alpha$, and choose $\alpha > 0$ so small that*

$$r(\{e^{-\alpha} \leq |z| \leq e^\alpha\}) \subset \{|z| < e^t\}.$$

This allows Lemma 5.8 to apply at each slicing step.

Choose a decomposition

$$\tau = \mu_1 + \cdots + \mu_k \quad \text{with } 0 < \mu_j < \alpha.$$

Set $q_0 = r$ and $t_0 = t$. For each $j \geq 1$, define

$$\rho_{j-1} = R(it_{j-1}) \circ q_{j-1} \circ R(i\mu_j),$$

let $(p_j, R(it_j), q_j)$ be the canonical decomposition of ρ_{j-1} , and finally set

$$\Pi(R(it), r, R(i\tau)) = (p_1 p_2 \cdots p_k, R(it_k), q_k).$$

The key point is that the output does not depend on the chosen partition, once the pieces are small. Neretin proves this using analytic dependence of canonical conformal maps on parameters.

Lemma 5.10 (Real-analyticity of one slicing output). *Fix a choice of a decomposition $\tau = \mu_1 + \cdots + \mu_k$ with $0 < \mu_j < \alpha$. Assume the slicing procedure in Definition 5.9 is defined for the given data. Then the resulting triple $\Pi(R(it), r, R(i\tau))$ depends real-analytically on the parameters.*

Proof. Each slicing step is built from a canonical uniformization of a two-connected domain by a round annulus, together with composition of holomorphic maps. Neretin's Lemma 2 says that the canonical uniformization varies real-analytically with parameters. Composing real-analytic families of holomorphic maps keeps real-analytic dependence. So the final output varies real-analytically as well. $\square$

Lemma 5.11 (Agreement for small τ). *Fix $t > 0$ and $r \in \text{Diff}^+(S^1)$ as in Definition 5.9. There is $\tau_0 > 0$ such that the following holds for every $0 < \tau < \tau_0$. For any two decompositions*

$$\tau = \mu_1 + \cdots + \mu_k = \nu_1 + \cdots + \nu_\ell \quad (0 < \mu_j, \nu_j < \alpha),$$

the slicing procedure is defined, and both choices give the same output.

Proof. Take τ_0 so small that every intermediate collar insertion stays inside the annulus where r has a univalent continuation. Then Lemma 5.8 applies at each slicing step. So all intermediate identities hold on a common annulus neighbourhood of S^1 .

In this range, the slicing procedure produces a canonical decomposition of the single map $\rho = R(it) \circ r \circ R(i\tau)$. The canonical decomposition is unique by Lemma 5.6. So the output does not depend on the chosen decomposition of τ . $\square$

Lemma 5.12 (Uniqueness for real-analytic families). *Let $U \subset \mathbb{R}^m$ be connected and open. Let $F, G : U \rightarrow X$ be real-analytic maps into a finite-dimensional real-analytic manifold X . If $F = G$ on a nonempty open subset of U , then $F = G$ on all of U .*

Proof. Work in one real-analytic chart of X . In coordinates, each component of $F - G$ is a real-analytic function on U . A real-analytic function that is zero on a nonempty open set is zero everywhere on a connected domain. So $F = G$ on U . $\square$

Lemma 5.13 (Correctness of Π). *Under the assumptions of Definition 5.9, the element $\Pi(R(it), r, R(i\tau))$ is independent of the chosen decomposition $\tau = \mu_1 + \cdots + \mu_k$ with $0 < \mu_j < \alpha$. Moreover, the output depends real-analytically on the input parameters.*

Proof. Fix two decompositions of the same τ , $\tau = \mu_1 + \cdots + \mu_k = \nu_1 + \cdots + \nu_\ell$, with all parts $< \alpha$. Let F be the output produced using the μ -decomposition. Let G be the output produced using the ν -decomposition.

By Lemma 5.10, both F and G depend real-analytically on the parameters. In particular, for fixed t and r , they are real-analytic functions of τ on any connected interval where the slicing procedure is defined.

By Lemma 5.11, there is an interval $(0, \tau_0)$ on which $F(\tau) = G(\tau)$. By Lemma 5.12, the equality extends to the whole connected domain of definition. So the output does not depend on the chosen decomposition of τ . Real-analytic dependence is from Lemma 5.10. $\square$

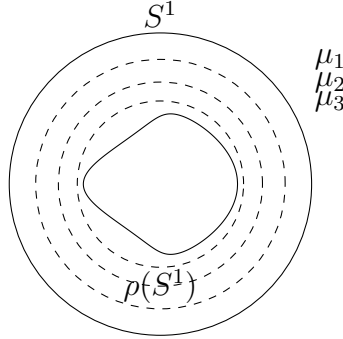


Figure 3: Π slices $\tau = \mu_1 + \mu_2 + \mu_3 + \cdots$, repeatedly re-uniformizes, and records new boundary maps.

Definition 5.14 (Multiplication operator for the semigroup). *Let*

$$\rho_1 = (p_1, R(it_1), q_1), \quad \rho_2 = (p_2, R(it_2), q_2).$$

Apply the exhaustion procedure to the middle triple

$$(R(it_1), q_1 \circ p_2, R(it_2))$$

and write

$$\Pi(R(it_1), q_1 p_2, R(it_2)) = (u, R(is), v).$$

Define

$$\rho_1 \rho_2 = (p_1 u, R(is), v q_2).$$

This is Neretin's multiplication formula.

5.5 Some consequences about the Virasoro algebra

We state a result from [Ner87] without proof.

Certain projective unitary representations of $\text{Diff}^+(S^1)$ extend to the annulus semigroup. The modulus $t > 0$ acts by a contraction semigroup. For more details about the Virasoro algebra see Appendix A.

Theorem 5.15 (Extension to the annulus semigroup). *Let π be the unitary projective representation of $\text{Diff}^+(S^1)$ coming from a unitarizable highest weight Virasoro representation (or a finite direct sum of such). Let Γ be Neretin's annulus semigroup. Write $\gamma \in \Gamma$ in canonical form $\gamma = (p, R(it), q)$.*

Let A_0 be the self-adjoint operator corresponding to the Virasoro generator u_0 in Neretin's notation. Set $L_0 := -A_0$. Define

$$\tilde{\pi}(\gamma) := \pi(p) \exp(-tL_0) \pi(q).$$

Then $\tilde{\pi}$ is a projective representation of Γ . For $t > 0$, the operator $\tilde{\pi}(\gamma)$ is trace class and contractive.

6 Further reading: Segals definition of CFT

The picture of dynamics built so far naturally leads into the study of conformal field theory (CFT) due to Segals definitions of CFT as a cobordism category. In this final section we only sketch out this idea following [Seg88].

Begin by recalling the notion of a cobordism.

Definition 6.1 (Cobordism). *For two $(n-1)$ dimensional manifolds M and N a choice of an n dimensional manifold C such that boundary of C is the union of M and N is called a cobordism from M to N . With abuse of notation we denote them as $C : M \rightarrow N$.*

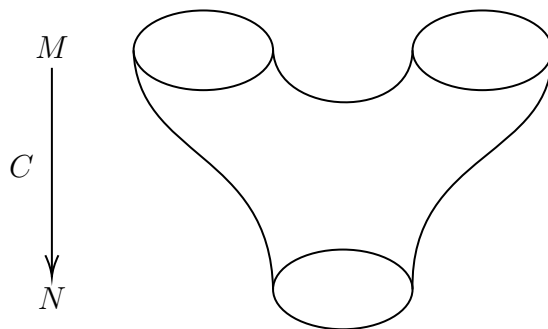


Figure 4: Pair of pants cobordism

We may interpret M, N as space at a particular time and $C : M \rightarrow N$ as the process of time evolution from M to N .

Recall that Neretin constructs a complex semigroup containing $\text{Diff}^+(S^1)$ (even though a complexified group does not exist).

6.1 The conformal cobordism category

Definition 6.2 (Segal's conformal category). *Let $\mathcal{C}$ be the category whose objects are C_n ($n \geq 0$), interpreted as n ordered copies of S^1 . A morphism $X : C_m \rightarrow C_n$ is an isomorphism class of compact Riemann surfaces X with boundary, together with:*

- *an ordered decomposition of ∂X into m incoming boundary circles and n outgoing boundary circles,*
- *for each boundary component, a real-analytic parametrization by S^1 (orientation-preserving for outgoing circles and orientation-reversing for incoming circles).*

Composition is defined by gluing: glue an outgoing boundary circle of one surface to the corresponding incoming boundary circle of the next, using the parametrizations. The monoidal structure is disjoint union.

One component of the space of cylinders $C_1 \rightarrow C_1$ is parametrized by a modulus q with $0 < |q| < 1$, and contains the standard annuli

$$A_q = \{z \in \mathbb{C} : |q| \leq |z| \leq 1\}.$$

Set

$$\mathcal{E} := \text{End}_{\mathcal{C}}(C_1).$$

For $q \in \mathbb{C}$ with $0 < |q| < 1$, let $P_q \in \mathcal{E}$ be the morphism represented by the annulus A_q with its two boundary circles parametrized by $e^{i\theta}$ on $|z| = 1$ and $qe^{i\theta}$ on $|z| = |q|$. Then $\{P_q\}$ is a subsemigroup of $\mathcal{E}$ and

$$P_{q_1} \circ P_{q_2} \cong P_{q_1 q_2}.$$

6.2 Segal's axioms

For a morphism $X : C_m \rightarrow C_n$, let $\bar{X} : C_n \rightarrow C_m$ be the same surface with complex-conjugate complex structure, with incoming and outgoing boundary circles swapped (so that it is a morphism $C_n \rightarrow C_m$).

Definition 6.3 (Conformal field theory (Segal)). *A conformal field theory is a representation of $\mathcal{C}$, namely a continuous functor $T : \mathcal{C} \rightarrow \mathbf{Hilb}$ to complex Hilbert spaces such that:*

1. $T(C_n) = H^{\otimes n}$ for some Hilbert space H (so $T(C_1) = H$).
2. For each morphism $X : C_m \rightarrow C_n$, the operator $T(X) : H^{\otimes m} \rightarrow H^{\otimes n}$ is trace-class.
3. T is a $*$ -functor: $T(X)^* = T(\bar{X})$, where $\bar{X}$ is the complex-conjugate Riemann surface.
4. (Crossing) If an incoming boundary circle is turned into an outgoing one by reversing orientation, the corresponding operators are adjoint in the natural identification of tensor products.
5. (Collapsing) If $X : C_{m+r} \rightarrow C_{n+r}$ is turned into $X' : C_m \rightarrow C_n$ by gluing the first r outgoing circles to the first r incoming circles, then

$$T(X') = \text{trace } T(X),$$

where the trace is taken over $H^{\otimes r}$.

6.3 Atiyah's axioms for TQFT

The following concise definition is reproduced as seen in [Bae04]. For the definitions related to monoidal categories refer to Appendix C.

A (Euclidean) n -dimensional topological quantum field theory is a symmetric monoidal functor

$$Z : \mathbf{nCob} \rightarrow \mathbf{Hilb},$$

where $\mathbf{nCob}$ has closed oriented $(n-1)$ -manifolds as objects and n -dimensional cobordisms as morphisms, with monoidal product given by disjoint union. The axioms are:

- $Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes Z(\Sigma_2)$ and $Z(\emptyset) \cong \mathbb{C}$.
- $Z(W_2 \circ W_1) = Z(W_2) \circ Z(W_1)$ and $Z(\text{id}_\Sigma) = \text{id}_{Z(\Sigma)}$.
- Orientation reversal corresponds to adjoints: $Z(\overline{W}) = Z(W)^*$.

In dimension 2, Segal's CFT is a refinement of this where cobordisms carry conformal structure and operators are required to be trace-class, with gluing implemented by taking traces.

Appendices

A Virasoro algebra

Definition A.1 (Lie algebra representation). *Let $\mathfrak{g}$ be a Lie algebra and V a complex vector space. A representation of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \text{End}(V)$.*

Definition A.2 (Witt algebra). *The Witt algebra is the complex Lie algebra with basis $(u_n)_{n \in \mathbb{Z}}$ and bracket*

$$[u_m, u_n] = (m - n)u_{m+n}.$$

It is the complexification of the Lie algebra of real-analytic vector fields on S^1 .

Definition A.3 (Central extension of a Lie algebra). *A central extension of a Lie algebra $\mathfrak{g}$ is a short exact sequence*

$$0 \longrightarrow \mathbb{C}z \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

such that z lies in the center of $\tilde{\mathfrak{g}}$, i.e. $[z, x] = 0$ for all x . The extension is one-dimensional if the kernel is $\mathbb{C}z$.

Definition A.4 (Virasoro algebra). *The Virasoro algebra V is the one-dimensional central extension of the Witt algebra with central element z and relations*

$$[u_m, u_n] = (m - n)u_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} z, \quad [u_n, z] = 0.$$

Definition A.5 (Projective unitary representation). *Let G be a group and $\mathcal{H}$ a Hilbert space. A projective unitary representation is a map $\pi : G \rightarrow U(\mathcal{H})$ such that*

$$\pi(g)\pi(h) = c(g, h) \pi(gh)$$

for some function $c : G \times G \rightarrow S^1$. The function c satisfies the cocycle identity coming from associativity.

Definition A.6 (Unitary representation of V). *A representation of the Virasoro algebra V on a pre-Hilbert space L is called unitary if there is a common invariant dense domain on which*

$$u_n^* = u_{-n}, \quad z^* = z,$$

and z acts by a real scalar.

Definition A.7 (Highest weight vector and highest weight module). *A vector $v_0 \neq 0$ is called a highest weight vector if*

$$u_n v_0 = 0 \quad (n > 0), \quad u_0 v_0 = h v_0, \quad z v_0 = c v_0$$

for some scalars h, c . A highest weight module is a V -module generated by such a vector.

Definition A.8 (Unitarizable highest weight representation). *A highest weight representation is unitarizable if it admits a positive definite Hermitian form for which the representation is unitary in the sense above.*

Definition A.9 (Rotations and the energy operator). *Let $r_\theta \in \text{Diff}^+(S^1)$ be the rotation $z \mapsto e^{i\theta}z$. In a positive energy projective unitary representation one can choose phases on the rotation subgroup so that $\theta \mapsto \pi(r_\theta)$ is a one-parameter unitary group. By Stone's theorem there is a self-adjoint operator L_0 such that*

$$\pi(r_\theta) = \exp(i\theta L_0).$$

The operator L_0 is called the energy operator.

Definition A.10 (Positive energy). *A projective unitary representation of $\text{Diff}^+(S^1)$ is called positive energy if the self-adjoint operator L_0 is bounded below. In an irreducible highest weight representation, the spectrum is of the form $h + \mathbb{Z}^+$ with finite-dimensional eigenspaces.*

Definition A.11 (Trace class operator). *Let T be a bounded operator on a Hilbert space. It is trace class if $\sum_k \langle (T^*T)^{1/2} e_k, e_k \rangle < \infty$ for some orthonormal basis $\{e_k\}$.*

B Functional analysis

The purpose of establishing this functional analysis machinery is primarily to extend our notion of exponentials to not necessarily finite operators. This is needed for the proof of Stone's theorem.

Our goal in this appendix is to present some useful tests for adjointness and a self contained proof of the spectral theorem using Gelfand-Naimark.

B.1 Convergence in $\mathcal{H}$

Let $\mathcal{H}$ denote a complex Hilbert space (complete inner product space).

The inner product defines a norm in the usual sense. We refer to [Hal13, Tes09] for the next following section.

Definition B.1 (Strong operator convergence). *Let (A_n) be operators on $\mathcal{H}$ and let A be an operator on $\mathcal{H}$. We say A_n converges strongly to A if*

$$\|A_n v - A v\| \rightarrow 0 \quad \text{for every } v \in \mathcal{H}.$$

We write $A_n \xrightarrow{s} A$.

Definition B.2 (Weak operator convergence). *Let (A_n) be operators on $\mathcal{H}$ and let A be an operator on $\mathcal{H}$. We say A_n converges weakly to A if*

$$\langle A_n v, w \rangle \rightarrow \langle A v, w \rangle \quad \text{for every } v, w \in \mathcal{H}.$$

We write $A_n \xrightarrow{w} A$.

The naming convention is due to the following fact.

Proposition B.3. *Strong operator convergence implies weak operator convergence.*

Proof. Assume $A_n \xrightarrow{s} A$. Fix $v, w \in \mathcal{H}$. Then

$$\langle A_n v, w \rangle - \langle A v, w \rangle = \langle (A_n - A)v, w \rangle$$

and hence

$$|\langle A_n v, w \rangle - \langle A v, w \rangle| \leq \|(A_n - A)v\| \|w\| \rightarrow 0.$$

Therefore, $A_n \xrightarrow{w} A$. □

Example B.4 (Weak convergence need not imply strong convergence). *Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and let U be the bilateral shift defined by $(Ux)_k = x_{k-1}$. Then U is unitary and $\|U^n x\| = \|x\|$ for every n and every x . For $x, y \in \ell^2(\mathbb{Z})$ one has*

$$\langle U^n x, y \rangle = \sum_{k \in \mathbb{Z}} x_{k-n} \overline{y_k} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence $U^n \xrightarrow{w} 0$ while U^n does not converge strongly to 0 since $\|U^n x\| = \|x\|$ for all n .

Lemma B.5 (Weak implies strong for unitary families). *Let $U(t)$ be unitary for each t and assume $t \mapsto \langle U(t)u, v \rangle$ is continuous at t_0 for all $u, v \in \mathcal{H}$. Then $t \mapsto U(t)v$ is norm continuous at t_0 for every $v \in \mathcal{H}$.*

Proof. It is enough to treat $t_0 = 0$. For $v \in \mathcal{H}$,

$$\|U(t)v - v\|^2 = \langle U(t)v - v, U(t)v - v \rangle = 2\|v\|^2 - 2\operatorname{Re}\langle U(t)v, v \rangle.$$

We know $t \mapsto \langle U(t)v, v \rangle$ is continuous at 0 and equals $\|v\|^2$ at $t = 0$. Hence the right hand side tends to 0 and $\|U(t)v - v\| \rightarrow 0$. For a general t_0 use $U(t) = U(t - t_0)U(t_0)$. □

B.2 Symmetric and self adjoint operators, extensions

Let $B(\mathcal{H})$ denote the space of bounded operators on $\mathcal{H}$. For an operator A (not necessarily bounded) we denote its domain as $\text{Dom}(A)$.

Definition B.6 (Adjoint of a densely defined operator). *Let A be a densely defined linear operator on $\mathcal{H}$. A vector y lies in $\text{Dom}(A^*)$ if there exists a vector $z \in \mathcal{H}$ such that*

$$\langle Ax, y \rangle = \langle x, z \rangle \quad \text{for all } x \in \text{Dom}(A).$$

*In that case we set $A^*y = z$. The operator A^* is called the adjoint of A .*

Definition B.7 (Symmetric). *A densely defined operator A on $\mathcal{H}$ is symmetric if*

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \text{for all } x, y \in \text{Dom}(A).$$

Lemma B.8. *If A is symmetric then $A \subset A^*$.*

Proof. Let $y \in \text{Dom}(A)$. For each $x \in \text{Dom}(A)$ symmetry gives

$$\langle Ax, y \rangle = \langle x, Ay \rangle.$$

The map $x \mapsto \langle Ax, y \rangle$ is bounded on $\text{Dom}(A)$ because

$$|\langle Ax, y \rangle| = |\langle x, Ay \rangle| \leq \|x\| \|Ay\|.$$

Since $\text{Dom}(A)$ is dense, $y \in \text{Dom}(A^*)$ and $A^*y = Ay$. Hence $A \subset A^*$. $\square$

Definition B.9 (Extension and symmetric extension). *Let A and B be densely defined operators on $\mathcal{H}$. We write $A \subset B$ if $\text{Dom}(A) \subset \text{Dom}(B)$ and $Bx = Ax$ for all $x \in \text{Dom}(A)$. If both A and B are symmetric and $A \subset B$, then B is called a symmetric extension of A .*

Proposition B.10. *Self adjoint operators have no proper symmetric extensions. More precisely, if A is self adjoint and B is symmetric with $A \subset B$, then $B = A$.*

Proof. We show first that $B \subset A^*$. Let $y \in \text{Dom}(B)$. Because $A \subset B$, one has $\text{Dom}(A) \subset \text{Dom}(B)$ and $Bx = Ax$ for all $x \in \text{Dom}(A)$. Using symmetry of B we compute for $x \in \text{Dom}(A)$:

$$\langle Ax, y \rangle = \langle Bx, y \rangle = \langle x, By \rangle.$$

The map $x \mapsto \langle Ax, y \rangle$ is bounded on $\text{Dom}(A)$ since

$$|\langle Ax, y \rangle| = |\langle x, By \rangle| \leq \|x\| \|By\|.$$

Because $\text{Dom}(A)$ is dense, this shows $y \in \text{Dom}(A^*)$ and $A^*y = By$. Therefore $B \subset A^*$.

Now use that A is self adjoint, so $A^* = A$. Hence $B \subset A$. Together with $A \subset B$ we conclude $A = B$. $\square$

Definition B.11 (Closed operator and closure). *Let A be a densely defined operator on $\mathcal{H}$. Its graph is*

$$\mathcal{G}(A) := \{(v, Av) \in \mathcal{H} \times \mathcal{H} : v \in \text{Dom}(A)\}.$$

We say A is closed if $\mathcal{G}(A)$ is closed in $\mathcal{H} \times \mathcal{H}$.

Equivalently A is closed if for all $v_n \in \text{Dom}(A)$, $v_n \rightarrow v$, and $Av_n \rightarrow w$, then $v \in \text{Dom}(A)$ and $Av = w$.

If A admits a closed extension, its closure $\overline{A}$ is the minimal closed operator with $A \subset \overline{A}$.

Definition B.12 (Self adjoint and essentially self adjoint). *A densely defined A is self adjoint if $A = A^*$. A symmetric A is essentially self adjoint if its closure $\overline{A}$ is self adjoint.*

We can now see a simple adjointness criterion.

Lemma B.13. *Let A be densely defined symmetric on $\mathcal{H}$. If there exists $z \in \mathbb{C}$ such that $\text{Ran}(A + z) = \text{Ran}(A + z^*) = \mathcal{H}$ then A is self-adjoint.*

Proof. Let $\psi \in \text{Dom}(A^*)$ and $A^*\psi = \tilde{\psi}$. Since $\text{Ran}(A + z^*) = \mathcal{H}$, pick $\vartheta \in \text{Dom}(A)$ such that $(A + \bar{z})\vartheta = \tilde{\psi} + \bar{z}\psi$. For $\varphi \in \text{Dom}(A)$,

$$\langle \psi, (A + z)\varphi \rangle = \langle \tilde{\psi} + z^*\psi, \varphi \rangle = \langle (A + z^*)\vartheta, \varphi \rangle = \langle \vartheta, (A + z)\varphi \rangle.$$

Since $\text{Ran}(A + z) = \mathcal{H}$, we get $\psi = \vartheta \in \text{Dom}(A)$. Thus $\text{Dom}(A^*) \subset \text{Dom}(A)$ and $A = A^*$. $\square$

This can be generalized into the following form which we will use in the proof of Stone's theorem.

Lemma B.14 (Adjointness criterion). *Let A be densely defined and symmetric and let $z \in \mathbb{C} \setminus \mathbb{R}$. The following are equivalent:*

1. $\text{Ran}(A + z) = \mathcal{H}$ and $\text{Ran}(A + z^*) = \mathcal{H}$,
2. $\ker(A^* + z) = \{0\}$ and $\ker(A^* + z^*) = \{0\}$.

If these hold for one such z , then A is essentially self-adjoint.

Proof. (1) $\iff$ (2): For $T = A + z$ we have $\overline{\text{Ran}(T)} = (\ker T^*)^\perp$ and $(A + z)^* = A^* + z^*$; the same identity with z^* gives the equivalence.

Assume (2). Since A is densely defined, $\overline{A}$ is symmetric and $(\overline{A})^* = A^*$, so (2) holds for $\overline{A}$ as well. Replacing A by $\overline{A}$ does not change (2), and a closed operator is convenient for the range argument, so we may assume A is closed.

Write $z = x + iy$ with $y \neq 0$. For any $\psi \in \text{Dom}(A)$,

$$\|(A + z)\psi\|^2 = \|(A + x)\psi + iy\psi\|^2 = \|(A + x)\psi\|^2 + y^2\|\psi\|^2,$$

because A is symmetric and $\langle (A + x)\psi, \psi \rangle \in \mathbb{R}$, hence the cross term has real part 0. It follows that $\|(A + z)\psi\| \geq |y| \|\psi\|$. Therefore $\ker(A + z) = \{0\}$ and $\text{Ran}(A + z)$ is closed: if $(A + z)\psi_n \rightarrow v$ then the inequality makes $\{\psi_n\}$ Cauchy in $\text{Dom}(A)$, so $\psi_n \rightarrow \psi$ and $v = (A + z)\psi$.

From (2) we have $\ker(A^* + z^*) = \{0\}$, hence

$$\overline{\text{Ran}(A + z)} = (\ker(A^* + z^*))^\perp = \mathcal{H}.$$

Since the range is both dense and closed, $\text{Ran}(A + z) = \mathcal{H}$. The same argument with z^* gives $\text{Ran}(A + z^*) = \mathcal{H}$. By Lemma B.13 the (closed) operator A is self-adjoint, so the original operator is essentially self-adjoint.

Conversely, assume A is self-adjoint. The identity above still holds, so $\ker(A + z) = \{0\}$ and $\text{Ran}(A + z)$ is closed. If $u \perp \text{Ran}(A + z)$, then for all $\psi \in \text{Dom}(A)$,

$$0 = \langle u, (A + z)\psi \rangle = \langle (A + z^*)u, \psi \rangle,$$

so $(A + z^*)u = 0$. Taking the inner product with u and using $\langle Au, u \rangle \in \mathbb{R}$ gives

$$0 = \langle (A + z^*)u, u \rangle = \langle Au, u \rangle + z^*\|u\|^2,$$

whose imaginary part is $y\|u\|^2 = 0$. Thus $u = 0$, so $\text{Ran}(A + z)^\perp = \{0\}$ and therefore $\text{Ran}(A + z) = \mathcal{H}$. The same argument with z^* yields $\text{Ran}(A + z^*) = \mathcal{H}$, which is (1). $\square$

B.3 Projection valued measures

We now set up the machinery required for the spectral theorems.

Definition B.15 (Resolvent and spectrum). *Let A be a densely defined linear operator on $\mathcal{H}$. The resolvent set $\rho(A)$ consists of all $z \in \mathbb{C}$ such that $A - z$ is bijective with bounded inverse on $\mathcal{H}$*

The spectrum is $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Example B.16. *We list a few examples of resolvents and spectrums.*

- For $A = \text{diag}(\lambda_1, \dots, \lambda_n)$,

$$(\lambda I - A)^{-1} = \text{diag}\left(\frac{1}{\lambda - \lambda_1}, \dots, \frac{1}{\lambda - \lambda_n}\right)$$

when $\lambda \notin \{\lambda_1, \dots, \lambda_n\}$. So $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$.

- If A is self adjoint, i.e., $A = A^*$. Then for $z = x + iy$ with $y \neq 0$,

$$\|(A - z)\psi\|^2 = \|(A - x)\psi\|^2 + y^2\|\psi\|^2,$$

so $z \in \rho(A)$. Hence $\sigma(A) \subset \mathbb{R}$.

- Let $M : L^2(0, 1) \rightarrow L^2(0, 1)$ defined as the left multiplication operator

$$(Mf)(x) = x f(x).$$

If $\lambda \notin [0, 1]$, then $(M - \lambda)^{-1}f(x) = \frac{1}{x - \lambda}f(x)$ and $\|(M - \lambda)^{-1}\| = 1/\text{dist}(\lambda, [0, 1])$. For $\lambda \in [0, 1]$, the multiplier $\frac{1}{x - \lambda} \notin L^\infty(0, 1)$, so $(M - \lambda)^{-1}$ is unbounded. Hence $\sigma(M) = [0, 1]$.

We can now introduce a very important notion of a projection valued measure.

Definition B.17 (Projection valued measure). *Let Ω be a σ -algebra on a set X . A map $\mu : \Omega \rightarrow B(\mathcal{H})$ is said to be a projection valued measure (PVM) if*

1. $\mu(A) = \mu(A)^*$ and $\mu(A)^2 = \mu(A)$ for all $A \in \Omega$.
2. $\mu(\emptyset) = 0, \mu(X) = I$.
3. For pairwise disjoint $\{A_k\}_{k=1}^\infty$ and all $v \in \mathcal{H}$, $\mu(\cup_{k=1}^\infty A_k)v = \sum_{k=1}^\infty \mu\{A_k\}v$.

We will see some examples of PVM's after we have defined operator valued integration against a PVM.

Corollary B.18. *If μ is a PVM then $\mu(A)\mu(B) = \mu(A \cap B) = \mu(B)\mu(A)$ for all $A, B \in \Omega$.*

Proof. Write $A = (A \cap B) \cup (A \setminus B)$ and $B = (A \cap B) \cup (B \setminus A)$, where the unions are disjoint. By the countable additivity of μ ,

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B), \quad \mu(B) = \mu(A \cap B) + \mu(B \setminus A).$$

Since the sets $A \cap B$, $A \setminus B$, and $B \setminus A$ are pairwise disjoint, the corresponding projections are mutually orthogonal, i.e.,

$$\mu(A \cap B)\mu(A \setminus B) = 0, \quad \mu(A \cap B)\mu(B \setminus A) = 0, \quad \mu(A \setminus B)\mu(B \setminus A) = 0.$$

Therefore,

$$\begin{aligned} \mu(A)\mu(B) &= (\mu(A \cap B) + \mu(A \setminus B))(\mu(A \cap B) + \mu(B \setminus A)) \\ &= \mu(A \cap B)^2 + \mu(A \cap B)\mu(B \setminus A) + \mu(A \setminus B)\mu(A \cap B) \\ &\quad + \mu(A \setminus B)\mu(B \setminus A) \\ &= \mu(A \cap B) + 0 + 0 + 0 \\ &= \mu(A \cap B). \end{aligned}$$

Similarly, $\mu(B)\mu(A) = \mu(A \cap B)$. □

A naïve thought would be to inquire why property 3 of PVM couldn't be phrased at the operator level. However, this fails in the most natural case of $L^2(\mathbb{R})$ as seen below.

Example B.19. *Let $\mathcal{H} = L^2(\mathbb{R})$, and let Ω be the Borel σ -algebra on $\mathbb{R}$. Define the projection-valued measure $\mu : \Omega \rightarrow \mathcal{B}(\mathcal{H})$ by*

$$(\mu(A)f)(x) = \chi_A(x)f(x), \quad \text{for all } f \in L^2(\mathbb{R}), \quad A \in \Omega,$$

where χ_A denotes the characteristic function of the set A . Then μ is a projection-valued measure.

Proof. For $f, g \in L^2(\mathbb{R})$,

$$\mu(A)^2 f = \chi_A^2 f = \chi_A f \quad \Rightarrow \quad \mu(A)^2 = \mu(A),$$

$$\langle \mu(A)f, g \rangle = \int \chi_A f \bar{g} = \int f \overline{\chi_A g} = \langle f, \mu(A)g \rangle \Rightarrow \mu(A)^* = \mu(A).$$

Also $\mu(\emptyset) = 0$, $\mu(\mathbb{R}) = I$, and

$$\mu(A)\mu(B)f = \chi_A \chi_B f = \chi_{A \cap B} f = \mu(A \cap B)f.$$

If A_k are pairwise disjoint and $A = \bigcup_{k \geq 1} A_k$, then with $S_N = \sum_{k=1}^N \mu(A_k)$,

$$\|\mu(A)f - S_N f\|_2^2 = \int_{\bigcup_{k > N} A_k} |f|^2 \xrightarrow{N \rightarrow \infty} 0,$$

so $S_N f \rightarrow \mu(A)f$ in L^2 . Hence μ is a PVM.

Why strong additivity is the right notion Let $\mathcal{H} = L^2(\mathbb{R})$ and let $\mu(E)$ be multiplication by the indicator χ_E . If $E_k = [k, k+1)$ then for every $f \in L^2(\mathbb{R})$ one has $\sum_{k=1}^N \mu\{e_k\}f \rightarrow \mu([1, \infty))f$ in L^2 . The operator norms do not converge because

$$\left\| \sum_{k=1}^N \mu\{e_k\} - \mu([1, \infty)) \right\| = \|\mu([N+1, \infty))\| = 1 \quad \text{for all } N.$$

Therefore requiring operator norm additivity would fail even in this basic example

Proposition B.20 (Operator-valued integration against a PVM). *Let (X, Ω) be a measurable space and let $\mu : \Omega \rightarrow B(\mathcal{H})$ be a projection-valued measure. If $f : X \rightarrow \mathbb{C}$ is bounded and measurable, there exists a unique $T_f \in B(\mathcal{H})$ such that for all $v, w \in \mathcal{H}$ and $\mu_{w,v}(E) := \langle w, \mu(E)v \rangle$*

$$\langle w, T_f v \rangle = \int_X f d\mu_{w,v}.$$

Additionally the following properties hold

1. $T_{fg} = T_f T_g$ and $T_{\bar{f}} = T_f^*$ for all bounded measurable f, g
2. $\|T_f\| \leq \|f\|_\infty$
3. If $\mu = \mu^F$ is the spectral PVM of self-adjoint F , then $\|T_f\| = \|f\|_{L^\infty(\sigma(F))}$.

Proof. We first prove it for simple functions then pass to the limit.

Simple f . Let $f = \sum_{j=1}^m \alpha_j \mathbf{1}_{B_j}$ with B_j pairwise disjoint. Define $T_f = \sum_{j=1}^m \alpha_j \mu(B_j)$. For $v, w \in \mathcal{H}$,

$$\langle w, T_f v \rangle = \sum_j \alpha_j \langle w, \mu(B_j)v \rangle = \int f d\mu_{w,v}.$$

1. $T_{fg} = T_f T_g$ for simple $g = \sum_k \beta_k \mathbf{1}_{C_k}$

$$T_f T_g = \sum_{j,k} \alpha_j \beta_k \mu(B_j) \mu(C_k) = \sum_{j,k} \alpha_j \beta_k \mu(B_j \cap C_k) = T_{fg}.$$

2. $T_{\bar{f}} = T_f^*$

$$T_f^* = \left(\sum_j \alpha_j \mu(B_j) \right)^* = \sum_j \overline{\alpha_j} \mu(B_j) = T_{\bar{f}}.$$

3. $\|T_f\| \leq \|f\|_\infty$ since $\mu(B_j) \mu(B_k) = 0$ for $j \neq k$,

$$\begin{aligned} \|T_f v\|^2 &= \left\| \sum_j \alpha_j \mu(B_j) v \right\|^2 = \sum_j |\alpha_j|^2 \|\mu(B_j) v\|^2 \leq \|f\|_\infty^2 \sum_j \|\mu(B_j) v\|^2 \\ &\leq \|f\|_\infty^2 \|v\|^2. \end{aligned}$$

Uniqueness for simple f : if $S \in B(\mathcal{H})$ satisfies $\langle w, Sv \rangle = \int f d\mu_{w,v}$ for all v, w , then $\langle w, (S - T_f)v \rangle = 0$ for all v, w , hence $S = T_f$.

Bounded measurable f . Choose simple $f_n \rightarrow f$ uniformly and set $T_f = \lim_n T_{f_n}$ in operator norm. Then

$$\langle w, T_f v \rangle = \lim_n \langle w, T_{f_n} v \rangle = \lim_n \int f_n d\mu_{w,v} = \int f d\mu_{w,v}.$$

The bounds $\|T_f\| \leq \|f\|_\infty$, $T_{\bar{f}} = T_f^*$, and $T_{fg} = T_f T_g$ follow by passing limits from the simple case. Uniqueness as above.

Spectral case. Let $\mu = \mu^F$ be the spectral PVM of a self adjoint F . The upper bound $\|T_f\| \leq \|f\|_\infty$ holds from above. For $\varepsilon > 0$ set $B_\varepsilon = \{\lambda: |f(\lambda)| > \|f\|_\infty - \varepsilon\}$. If $\mu(B_\varepsilon) \neq 0$, take $0 \neq v \in \text{Ran } \mu(B_\varepsilon)$ and $u = v/\|v\|$. Then

$$\|T_f u\|^2 = \langle u, T_{|f|^2} u \rangle = \int |f|^2 d\mu_{u,u} \geq (\|f\|_\infty - \varepsilon)^2.$$

Hence $\|T_f\| \geq \|f\|_\infty - \varepsilon$ for all $\varepsilon > 0$, so $\|T_f\| = \|f\|_\infty = \|f\|_{L^\infty(\sigma(F))}$. $\square$

Example B.21. We can now see some instructive examples for PVM's.

- Let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ on $\mathbb{C}^n$, $\lambda_j \in \mathbb{R}$. Define

$$E(B) = \text{diag}(\mathbf{1}_B(\lambda_1), \dots, \mathbf{1}_B(\lambda_n)).$$

Then E is a PVM and

$$A = \int_{\mathbb{R}} \lambda dE(\lambda).$$

- Let P be an orthogonal projection on $\mathcal{H}$. Put $A = I - P$. Then the spectrum is $\{0, 1\}$ and

$$E(\{0\}) = P, \quad E(\{1\}) = I - P.$$

Then

$$e^{itA} = P + e^{it}(I - P).$$

- Let $\mathcal{H} = L^2(X, \mu)$. Let $a : X \rightarrow \mathbb{R}$ be measurable. Define

$$(Af)(x) = a(x)f(x), \quad \text{Dom}(A) = \{f \in L^2 : af \in L^2\}.$$

Define

$$(E(B)f)(x) = \mathbf{1}_{\{x: a(x) \in B\}}(x) f(x).$$

B.4 Spectral theorem

B.5 C^* -algebras and the spectral theorem (bounded self-adjoint case)

We follow [Arv01, Chapters 1–3]. We use commutative Gelfand–Naimark without proof.

Definition B.22 ($*$ -algebra, C^* -algebra). A $*$ -algebra is a complex algebra A with a map $a \mapsto a^*$ such that $(ab)^* = b^*a^*$, $(a^*)^* = a$, and $(\lambda a + b)^* = \bar{\lambda}a^* + b^*$. A C^* -algebra is a Banach $*$ -algebra $(A, \|\cdot\|)$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Theorem B.23 (Gelfand–Naimark theorem (commutative case)). If A is a unital commutative C^* -algebra, then A is $*$ -isomorphic (isometrically) to $C(X)$ for a compact Hausdorff space X .

Lemma B.24 (Continuous functional calculus). Let $F \in B(\mathcal{H})$ be self-adjoint and put $X := \sigma(F)$. There is a canonical unital $*$ -homomorphism

$$\pi : C(X) \rightarrow B(\mathcal{H})$$

such that $\pi(\text{id}) = F$, where $\text{id}(\lambda) = \lambda$ on X . We write $\pi(f) = f(F)$.

Proof. Let $A_F = C^*(F, I) \subset B(\mathcal{H})$. This is a unital commutative C^* -algebra. By Theorem B.23, $A_F \simeq C(Y)$ for a compact Hausdorff space Y . Under this identification, F corresponds to a real-valued continuous function on Y . Its range equals $\sigma(F)$. Composing with the pullback along that function gives a unital $*$ -homomorphism $\pi : C(\sigma(F)) \rightarrow A_F \subset B(\mathcal{H})$. It sends id to F . $\square$

Definition B.25 ($*$ -representation, cyclic). *Let X be compact. A $*$ -representation is a unital $*$ -homomorphism $\rho : C(X) \rightarrow B(\mathcal{K})$. It is cyclic if there is $\xi \in \mathcal{K}$ such that $\overline{\rho(C(X))\xi} = \mathcal{K}$.*

Lemma B.26 (Cyclic model). *Let $\rho : C(X) \rightarrow B(\mathcal{K})$ be cyclic with cyclic vector ξ . Then there exists a finite regular Borel measure μ on X and a unitary $U : \mathcal{K} \rightarrow L^2(X, \mu)$ such that $U\xi = 1$ and*

$$U \rho(f) U^{-1} = M_f$$

for all $f \in C(X)$, where M_f is multiplication by f .

Proof. Define $\varphi(f) = \langle \rho(f)\xi, \xi \rangle$. This is positive. By Theorem ??, there is a finite regular Borel measure μ with $\varphi(f) = \int_X f d\mu$.

Define $T : C(X) \rightarrow \mathcal{K}$ by $T(f) = \rho(f)\xi$. Then

$$\|T(f)\|^2 = \langle \rho(\bar{f}f)\xi, \xi \rangle = \int_X |f|^2 d\mu.$$

So T is an isometry from $C(X)$ (with the $L^2(\mu)$ norm) into $\mathcal{K}$. It extends uniquely to an isometry $U_0 : L^2(X, \mu) \rightarrow \mathcal{K}$. Cyclicity says $T(C(X))$ is dense, so U_0 is surjective. Set $U = U_0^{-1}$. Then $U\xi = 1$. For $g \in C(X)$,

$$U \rho(f) U^{-1}(g) = U \rho(f) T(g) = U T(fg) = fg = M_f(g).$$

$\square$

Lemma B.27 (Decomposition into cyclic subrepresentations). *Let $\rho : C(X) \rightarrow B(\mathcal{H})$ be a $*$ -representation. Then there exist vectors $(\xi_j)_{j \in J} \subset \mathcal{H}$ such that*

$$\mathcal{H} = \bigoplus_{j \in J} \overline{\rho(C(X))\xi_j}.$$

Proof. Let $\mathcal{F}$ be the set of cyclic subspaces $M = \overline{\rho(C(X))\eta}$. Choose a maximal orthogonal family $\{M_j\}_{j \in J} \subset \mathcal{F}$ by Zorn. Let $M = \bigoplus_j M_j$. If $M^\perp \neq \{0\}$, pick $0 \neq \eta \in M^\perp$. Then $\overline{\rho(C(X))\eta} \subset M^\perp$ is a nonzero cyclic subspace orthogonal to every M_j . This contradicts maximality. So $M^\perp = \{0\}$. $\square$

Theorem B.28 (Spectral theorem for bounded self-adjoint operators). *If $F \in B(\mathcal{H})$ is self-adjoint then there exists a unique PVM μ^F on the Borel σ -algebra of $\sigma(F)$ such that*

$$F = \int_{\sigma(F)} \lambda d\mu^F(\lambda).$$

Proof. Put $X = \sigma(F)$. Let $\pi : C(X) \rightarrow B(\mathcal{H})$ be the continuous functional calculus from Lemma B.24. So $\pi(\text{id}) = F$.

First reduce to cyclic component. By Lemma B.27, there are closed subspaces $H_j = \overline{\pi(C(X))\xi_j}$ such that $\mathcal{H} = \bigoplus_{j \in J} H_j$. Each H_j is π -invariant. Let π_j be π restricted to H_j .

Next we construct a PVM on each cyclic component. Fix j . The representation π_j is cyclic with cyclic vector ξ_j . By Lemma B.26, there are a measure μ_j and a unitary $U_j : H_j \rightarrow L^2(X, \mu_j)$ such that $U_j \pi_j(f) U_j^{-1} = M_f$. For a Borel set $B \subset X$, define

$$\mu_j(B) = U_j^{-1} M_{1_B} U_j.$$

This is a PVM on H_j , because M_{1_B} is an orthogonal projection and it is strongly countably additive.

Now simply combine them together to define a global PVM. Define $\mu^F(B) = \bigoplus_{j \in J} \mu_j(B)$ on $\mathcal{H} = \bigoplus_j H_j$. Then μ^F is a PVM on X .

Using this we can now recover F . For $f \in C(X)$, we have on each H_j :

$$\pi_j(f) = U_j^{-1} M_f U_j = \int_X f(\lambda) d\mu_j(\lambda).$$

Taking the direct sum gives

$$\pi(f) = \int_X f(\lambda) d\mu^F(\lambda).$$

Now take $f = \text{id}$. Then

$$F = \pi(\text{id}) = \int_X \lambda d\mu^F(\lambda).$$

Lastly we check uniqueness. Let ν be a PVM on X such that $F = \int_X \lambda d\nu(\lambda)$. For $f \in C(X)$, define $S_f = \int_X f d\nu$ using Proposition B.20. Then $f \mapsto S_f$ is a unital $*$ -homomorphism on $C(X)$ and $S_{\text{id}} = F$. By uniqueness

of the continuous functional calculus, $S_f = f(F) = \pi(f)$ for all $f \in C(X)$. Hence for all $v, w \in \mathcal{H}$,

$$\int_X f d\nu_{w,v} = \langle w, S_f v \rangle = \langle w, \pi(f)v \rangle = \int_X f d(\mu^F)_{w,v}$$

for all $f \in C(X)$. By the scalar Riesz representation theorem, $\nu_{w,v} = (\mu^F)_{w,v}$ for all v, w . So $\nu(B) = \mu^F(B)$ for all Borel B . $\square$

C Monoidal categories

A monoidal category is a category $\mathbf{C}$ with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, a ‘unit’ element I , and natural isomorphisms that make the functor associative and unital with I as expected. It is a generalization of the notion of a ‘tensor product’. It is used to define enriched categories.

This can be formalized as follows, there exists $I \in \mathbf{C}$ and natural isomorphisms,

- $\alpha_{A,B,C} : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$ (read as associator)
- $\lambda_A : I \otimes A \rightarrow A$ (read as left unitor)
- $\rho_A : A \otimes I \rightarrow A$ (read as right unitor)

And the following diagrams commute,

$$\begin{array}{ccc}
 & (A \otimes B) \otimes (C \otimes D) & \\
 \alpha_{A,B,C \otimes D} \nearrow & & \searrow \alpha_{A \otimes B, C, D} \\
 A \otimes (B \otimes (C \otimes D)) & & ((A \otimes B) \otimes C) \otimes D \\
 1_A \otimes \alpha_{B,C,D} \downarrow & & \uparrow \alpha_{A,B,C} \otimes 1_D \\
 A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C, D}} & (A \otimes (B \otimes C)) \otimes D
 \end{array}$$

$$\begin{array}{ccc}
 A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B \\
 1_A \otimes \lambda_B \searrow & & \swarrow \rho_A \otimes 1_B \\
 & A \otimes B &
 \end{array}$$

Definition C.1 (Monoidal functor). *Let $(\mathbf{C}, \otimes, I)$ and $(\mathbf{D}, \otimes, I)$ be monoidal categories. A (strong) monoidal functor is a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ together with natural isomorphisms*

$$\phi_{A,B} : F(A) \otimes F(B) \rightarrow F(A \otimes B), \quad \phi_0 : I_{\mathbf{D}} \rightarrow F(I_{\mathbf{C}}),$$

such that F respects the associator and the unitors (the standard coherence diagrams commute).

Definition C.2 (Symmetric monoidal category). *A symmetric monoidal category is a monoidal category together with natural isomorphisms*

$$\beta_{A,B} : A \otimes B \rightarrow B \otimes A$$

such that $\beta_{B,A} \circ \beta_{A,B} = 1_{A \otimes B}$ and the hexagon coherence diagrams commute.

Definition C.3 (Symmetric monoidal functor). *Let $\mathbf{C}$ and $\mathbf{D}$ be symmetric monoidal categories. A monoidal functor $(F, \phi, \phi_0) : \mathbf{C} \rightarrow \mathbf{D}$ is symmetric monoidal if for all A, B*

$$\phi_{B,A} \circ \beta_{F(A), F(B)} = F(\beta_{A,B}) \circ \phi_{A,B}.$$

Example C.4. *In the context of this paper the canonical examples of monoidal categories are:*

- **Hilb** *the category of hilbert spaces with bounded linear operators.*
- **nCob** *the category of $(n-1)$ dimensions manifolds with n -cobordisms.*

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