Heller's Simple Proof of the Künneth Theorem

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March 24, 2026

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1 Introduction

We work first in $\mathbf{Ab} = \mathbb{Z}$-Mod. The target is the Künneth exact sequence

$$0 \to H(C) \otimes H(D) \to H(C \otimes D) \to \mathrm{Tor}_1^{\mathbb{Z}}(H(C), H(D)) \to 0,$$

and then the universal coefficient theorems. The proof follows Heller's short argument [1].

Two facts are special to $\mathbf{Ab}$.

(i) Every subgroup of a free abelian group is free.

(ii) Hence every abelian group has projective dimension at most 1.

Therefore $\operatorname{Tor}_i^{\mathbb{Z}}(-,-) = 0$ and $\operatorname{Ext}_{\mathbb{Z}}^i(-,-) = 0$ for $i \geq 2$.

2 Complexes

Definition 2.1. A graded abelian group is a family $C_\bullet = (C_n)_{n\in\mathbb{Z}}$. A graded map of degree r from $C_\bullet$ to $D_\bullet$ is a family of homomorphisms $f_n : C_n \to D_{n+r}$.

Definition 2.2. A chain complex $(C_\bullet, \varphi_\bullet)$ is a graded abelian group together with homomorphisms $\varphi_n : C_n \to C_{n-1}$ such that $\varphi_n\varphi_{n+1} = 0$ for all n. We write

$$Z_n(C) = \ker\varphi_n, \qquad B_n(C) = \mathrm{im}\varphi_{n+1}, \qquad H_n(C) = Z_n(C)/B_n(C).$$

Also $H(C) = \bigoplus_n H_n(C)$.

Definition 2.3. Let $(C_\bullet, \varphi_\bullet)$ and $(D_\bullet, \psi_\bullet)$ be chain complexes. A chain map $\alpha : C_\bullet \to D_\bullet$ is a graded map of degree 0 such that

$$\psi\alpha = \alpha\varphi.$$

If $\alpha, \beta : C_\bullet \to D_\bullet$ are chain maps, a chain homotopy from α to β is a graded map $h : C_\bullet \to D_\bullet$ of degree 1 such that

$$\alpha - \beta = \psi h + h\varphi.$$

In components,

$$\alpha_n - \beta_n = \psi_{n+1}h_n + h_{n-1}\varphi_n.$$

Definition 2.4. A short exact sequence of chain complexes is a sequence

$$0 \to A_\bullet \xrightarrow{\alpha} B_\bullet \xrightarrow{\beta} C_\bullet \to 0$$

which is exact in each degree.

For such a sequence there is a natural long exact sequence in homology

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_i(A) & \longrightarrow & H_i(B) & \longrightarrow & H_i(C) \\ & & & & & \overset{\delta_i}{\swarrow} & \\ H_{i-1}(A) & \longrightarrow & H_{i-1}(B) & \longrightarrow & H_{i-1}(C) & \longrightarrow & \cdots \end{array}$$

The connecting map is defined as usual. For $[c] \in H_i(C)$ choose $b \in B_i$ with $\beta(b) = c$. Then $\beta(\psi b) = 0$, so $\psi b \in \mathrm{im}\alpha$. Write $\psi b = \alpha(a)$. Then $[a] \in H_{i-1}(A)$ and $\delta_i[c] = [a]$.

3 Resolutions and derived functors

3.1 Projectives and resolutions

Definition 3.1. An object $P \in \mathbf{Ab}$ is projective if for every epimorphism $M \twoheadrightarrow N$ and every map $P \to N$ there is a lift $P \to M$. Dually, $I \in \mathbf{Ab}$ is injective if for every monomorphism $M \rightarrowtail N$ and every map $M \to I$ there is an extension $N \to I$.

Free abelian groups are projective.

Definition 3.2. A projective resolution of M is an exact complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

with each P_i projective. A free resolution is defined similarly.

Proposition 3.3. *Every abelian group admits a free resolution.*

Proof. Choose a surjection $F_0 \twoheadrightarrow M$ with F_0 free. Then choose a surjection $F_1 \twoheadrightarrow \ker(F_0 \to M)$ with F_1 free. Continue inductively. □

Proposition 3.4. *Let $F_\bullet \to M$ be a projective resolution and let $G_\bullet \to N$ be an exact complex with $H_0(G) \cong N$ and $H_i(G) = 0$ for $i > 0$. Any map $u : M \to N$ lifts to a chain map $\alpha : F_\bullet \to G_\bullet$, and any two such lifts are chain homotopic.*

Proof sketch. Lift $u \circ (F_0 \to M)$ through $G_0 \twoheadrightarrow N$ using projectivity of F_0. This gives α_0. Since $\alpha_0\varphi_1$ lands in $\ker(G_0 \to N) = \mathrm{im}\psi_1$, projectivity of F_1 gives α_1. Continue inductively.

For uniqueness up to homotopy, apply the same lifting argument to the difference of two lifts. Exactness of $G_\bullet$ gives maps $h_n : F_n \to G_{n+1}$ with $\alpha - \beta = \psi h + h\varphi$. □

Corollary 3.5. *Any two projective resolutions of the same module are chain homotopy equivalent.*

Remark 3.6. This is the basic reason resolutions are useful. A module M may be replaced by a projective resolution $F_\bullet \to M$. A map $M \to N$ is then represented by a homotopy class of chain maps between projective resolutions. The same idea, dually, uses injective resolutions.

3.2 Adjoints and exactness

Definition 3.7. A functor $L : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $R : \mathcal{D} \to \mathcal{C}$ if there are natural bijections

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, R(Y)).$$

Proposition 3.8 (Tensor–Hom adjunction)**.** *For abelian groups A, B, G there is a natural bijection*

$$\mathrm{Hom}(A \otimes B, G) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, G)).$$

Proof. Given $f : A \otimes B \to G$, define $\widetilde{f} : B \to \mathrm{Hom}(A, G)$ by $\widetilde{f}(b)(a) = f(a \otimes b)$. Given $g : B \to \mathrm{Hom}(A, G)$, define $\widehat{g} : A \otimes B \to G$ by $\widehat{g}(a \otimes b) = g(b)(a)$. These constructions are inverse. □

Proposition 3.9. *For fixed abelian groups A and G one has*

(i) $A \otimes -$ *is right exact*

(ii) $\mathrm{Hom}(A, -)$ *is covariant left exact*

(iii) $\mathrm{Hom}(-, G)$ *is contravariant left exact*

Proof. By adjunction, $A \otimes -$ is a left adjoint, hence preserves cokernels. Thus if $M' \to M \to M'' \to 0$ is exact, then

$$A \otimes M'' \cong \mathrm{coker}(A \otimes M' \to A \otimes M),$$

which is right exactness.

Also $\mathrm{Hom}(A, -)$ is a right adjoint, hence preserves kernels. This is covariant left exactness. For $\mathrm{Hom}(-, G)$, apply $\mathrm{Hom}(-, G)$ to $0 \to M' \to M \to M'' \to 0$ and read exactness directly from precomposition. □

3.3 Derived functors

Definition 3.10. Let F be an additive right exact covariant functor. If $P_\bullet \to M$ is a projective resolution, define

$$L_i F(M) = H_i(F P_\bullet).$$

Let T be an additive left exact covariant functor. If $M \to I^\bullet$ is an injective resolution, define

$$R^i T(M) = H^i(T I^\bullet).$$

For the contravariant left exact functor $\mathrm{Hom}(-, G)$, one may compute right derived functors from a projective resolution in the first variable.

Proposition 3.11. *Let F be an additive right exact covariant functor.*

(1) $L_i F$ *is resolution-invariant*

(2) $L_0 F \cong F$

(3) If P is projective then $L_i F(P) = 0$ for $i > 0$

(4) Every short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *induces a natural long exact sequence*

$$\begin{array}{ccccc} & & \cdots & \longrightarrow & L_iF(C) \\ & & & \xrightarrow{\delta_i} & \\ L_{i-1}F(A) & \longrightarrow & L_{i-1}F(B) & \longrightarrow & L_{i-1}F(C) \\ & & & \xrightarrow{\delta_{i-1}} & \\ L_{i-2}F(A) & \longrightarrow & \cdots & & \end{array}$$

The dual statements hold for right derived functors.

Proof sketch. Resolution invariance follows from Proposition 3.4 and Corollary 3.5. Different projective resolutions are chain homotopy equivalent, so after applying F they have isomorphic homology.

For L_0F, if $P_\bullet \to M$ is a projective resolution, then

$$H_0(FP_\bullet) = \operatorname{coker}(FP_1 \to FP_0) = F(M)$$

by right exactness of F.

If P is projective, use the resolution $\cdots \to 0 \to P \xrightarrow{1} P \to 0$.

For the long exact sequence, choose compatible projective resolutions, apply F, and then apply the long exact sequence in homology. For details see Weibel, Section 2.4. □

Remark 3.12. In **Ab**, free resolutions of length 1 suffice. If $0 \to R \to F \to M \to 0$ is exact with F free, then R is free. Hence $\operatorname{pd}_{\mathbb{Z}} M \le 1$. Therefore only L_0, L_1 and R^0, R^1 remain.

4 Tor and Ext

Definition 4.1. For $M, N \in \mathbf{Ab}$ define

$$\operatorname{Tor}_i^{\mathbb{Z}}(M,N) = L_i(M \otimes -)(N), \qquad \operatorname{Ext}^i_{\mathbb{Z}}(M,N) = R^i\operatorname{Hom}(M,-)(N).$$

If $0 \to R \xrightarrow{u} F \to M \to 0$ is a free resolution of length 1, then

$$\operatorname{Tor}_1^{\mathbb{Z}}(M,N) = \ker(u \otimes 1_N)$$

and

$$\operatorname{Ext}^1_{\mathbb{Z}}(M,N) = \operatorname{coker}\big(\operatorname{Hom}(F,N) \to \operatorname{Hom}(R,N)\big).$$

Corollary 4.2. *Every short exact sequence* $0 \to A' \to A \to A'' \to 0$ *in* **Ab** *yields natural exact sequences*

$$0 \to \operatorname{Tor}_1^{\mathbb{Z}}(A'',N) \to A' \otimes N \to A \otimes N \to A'' \otimes N \to 0$$

and

$$0 \to \mathrm{Hom}(A'', N) \to \mathrm{Hom}(A, N) \to \mathrm{Hom}(A', N) \to \mathrm{Ext}^1_{\mathbb{Z}}(A'', N)$$
$$\to \mathrm{Ext}^1_{\mathbb{Z}}(A, N) \to \mathrm{Ext}^1_{\mathbb{Z}}(A', N) \to 0.$$

Moreover $\mathrm{Tor}_i^{\mathbb{Z}}(-,-) = 0$ *and* $\mathrm{Ext}^i_{\mathbb{Z}}(-,-) = 0$ *for* $i \geq 2$.

Proposition 4.3. *For every abelian group* N *and integer* $n \geq 1$*,*

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, N) \cong N[n] := \{x \in N \mid nx = 0\}, \qquad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, N) \cong N/nN.$$

Proof. Use the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$. The first formula is the kernel of multiplication by n on N. The second is the cokernel. □

Definition 4.4. An extension of M by N is a short exact sequence

$$0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0.$$

Two extensions are equivalent if there is an isomorphism $\theta : E \to E'$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \cong\downarrow\theta & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

commutes.

Theorem 4.5. $\mathrm{Ext}^1_{\mathbb{Z}}(M, N)$ *is naturally identified with the set of equivalence classes of extensions of* M *by* N*.*

Proof sketch. Choose a free presentation $0 \to R \to F \to M \to 0$. Pull back an extension of M by N along $F \to M$. Since F is free, the pullback splits. A splitting gives a map $R \to N$, well-defined modulo maps extending across F. This yields an element of

$$\mathrm{coker}\big(\mathrm{Hom}(F, N) \to \mathrm{Hom}(R, N)\big) = \mathrm{Ext}^1_{\mathbb{Z}}(M, N).$$

Reversing the construction recovers the extension. For details see Weibel, Section 3.4. □

Example 4.6. Extensions of $\mathbb{Z}/n$ by N are classified by N/nN. If $e \in E$ maps to $\bar{1} \in \mathbb{Z}/n$, then $ne \in N$. Replacing e by $e + h$ changes ne by nh. Hence the class of ne in N/nN depends only on the extension class.

Proposition 4.7. *For integers* $m, n \geq 1$ *one has*

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m, n), \qquad \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m, n).$$

More generally, if R *is a ring and* $I, J \subseteq R$ *are ideals, then*

$$\mathrm{Tor}_1^R(R/I, R/J) \cong (I \cap J)/IJ.$$

If $x \in R$ *is not a zero-divisor, then*

$$\mathrm{Tor}_1^R(R/(x), M) \cong \{m \in M \mid xm = 0\}.$$

Proof sketch. The first two formulas follow from Proposition 4.3. For the ideal formula, tensor the exact sequence $0 \to I \to R \to R/I \to 0$ with R/J. Since $I \otimes_R R/J \cong I/IJ$, the kernel of $I/IJ \to R/J$ is $(I \cap J)/IJ$. The last formula is the special case $I = (x)$. □

Remark 4.8. For a group G and a left $\mathbb{Z}[G]$-module M, one has

$$H^n(G, M) \cong \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

So Ext also appears as group cohomology.

5 Künneth theorem

Definition 5.1. If $(C_\bullet, \varphi)$ and $(D_\bullet, \psi)$ are chain complexes, their tensor product complex is given by

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential

$$\partial(c \otimes d) = \varphi(c) \otimes d + (-1)^p c \otimes \psi(d) \qquad (c \in C_p).$$

For a complex C, the graded groups $Z(C)$, $B(C)$, and $H(C)$ are viewed as chain complexes with zero differential.

5.1 Prerequisite lemmas

Lemma 5.2. *If each C_n is free abelian, then each $Z_n(C)$ and $B_n(C)$ is free abelian.*

Proof. They are subgroups of free abelian groups. □

Lemma 5.3. *If $C_\bullet$ is degreewise free, then there is a short exact sequence of chain complexes*

$$0 \to Z(C) \xrightarrow{\iota} C \xrightarrow{\varphi} B(C)[1] \to 0,$$

where $B(C)[1]_n = B_{n-1}(C)$. This sequence splits degreewise.

Proof. Exactness is immediate. By Lemma 5.2, each $B_{n-1}(C)$ is free. Hence the surjection $\varphi_n : C_n \twoheadrightarrow B_{n-1}(C)$ has a section. □

Lemma 5.4 (Splitting lemma)**.** *Let*

$$0 \to A_\bullet \xrightarrow{\alpha} B_\bullet \xrightarrow{\beta} C_\bullet \to 0$$

be a short exact sequence of chain complexes. Suppose there is a degree 0 graded map $q : C_\bullet \to B_\bullet$ with $\beta q = 1$. Then

$$\varphi q - q\gamma = \alpha\theta$$

for a unique degree -1 graded map $\theta : C_\bullet \to A_\bullet$, and $H(\theta)$ is the connecting homomorphism

$$\delta : H(C) \to H(A).$$

Proof. Since $\beta(\varphi q - q\gamma) = 0$, the image of $\varphi q - q\gamma$ lies in $\ker\beta = \mathrm{im}\alpha$. Thus θ exists uniquely. If $c \in Z_n(C)$, then $q(c)$ is a lift of c and $\varphi q(c) = \alpha\theta(c)$. This is exactly the defining recipe for the connecting map. ☐

Lemma 5.5. *If F is a graded free abelian group with zero differential, then*

$$H_n(F \otimes D) \cong \bigoplus_{p+q=n} F_p \otimes H_q(D)$$

naturally in D.

Proof. Each F_p is free, hence flat. The differential on $F \otimes D$ acts only on the D-factor, so homology is computed degreewise. ☐

Lemma 5.6. *Assume $C_\bullet$ is degreewise free. Tensor the sequence in Lemma 5.3 with $D_\bullet$. The connecting map of*

$$0 \to Z(C) \otimes D \to C \otimes D \to B(C)[1] \otimes D \to 0$$

is induced by the inclusion $j : B(C)[1] \hookrightarrow Z(C)[1]$.

Proof. Choose a degreewise splitting $q : B(C)[1] \to C$ of φ. Then $\varphi q = j$. Now apply the Splitting lemma to $q \otimes 1_D$. ☐

5.2 Final proof

Theorem 5.7 (Künneth)**.** *Let $C_\bullet$ and $D_\bullet$ be bounded below chain complexes of abelian groups. Assume each C_n is free abelian. Then for each n there is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \xrightarrow{I} H_n(C \otimes D) \xrightarrow{K} \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C), H_q(D)) \to 0.$$

If both complexes are degreewise free, this sequence splits, not canonically.

Proof. Set

$$\mathrm{Tor}_1(H(C), H(D))_n = \bigoplus_{p+q=n} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C), H_q(D)).$$

By Lemma 5.3, there is a short exact sequence

$$0 \to Z(C) \xrightarrow{\iota} C \xrightarrow{\varphi} B(C)[1] \to 0.$$

Tensor with D to obtain

$$0 \to Z(C) \otimes D \xrightarrow{\iota \otimes 1} C \otimes D \xrightarrow{\varphi \otimes 1} B(C)[1] \otimes D \to 0.$$

Choose a degreewise splitting $q : B(C)[1] \to C$ of φ. By Lemma 5.6, the connecting map in homology is induced by $j : B(C)[1] \hookrightarrow Z(C)[1]$.

Since $Z(C)$ and $B(C)$ are graded free with zero differential, Lemma 5.5 identifies the homology triangle with

$$\begin{array}{ccc} B(C)\otimes H(D)_n & \xrightarrow{\;j\otimes 1\;} & Z(C)\otimes H(D)_n \\ & \searrow \quad \swarrow & \\ & H_n(C\otimes D) & \end{array}$$

This is Heller's triangle.

Now for each p and q, the short exact sequence

$$0 \to B_p(C) \xrightarrow{j} Z_p(C) \to H_p(C) \to 0$$

produces, after tensoring with $H_q(D)$, an exact sequence

$$0 \to \operatorname{Tor}_1^{\mathbb{Z}}(H_p(C), H_q(D)) \to B_p(C)\otimes H_q(D) \xrightarrow{j\otimes 1} Z_p(C)\otimes H_q(D) \to H_p(C)\otimes H_q(D) \to 0.$$

Summing over $p+q=n$ gives the exact row

$$\begin{array}{ccccccccccc} 0 & \to & \operatorname{Tor}_1(H(C), H(D))_n & \to & B(C)\otimes H(D)_n & \xrightarrow{j\otimes 1} & Z(C)\otimes H(D)_n & \to & (H(C)\otimes H(D))_n & \to & 0 \\ & & & & & \searrow \quad H_n(C\otimes D) \quad \swarrow & & & & & \end{array}$$

The row is exact. The triangle is exact. Therefore there are unique maps

$$I : (H(C)\otimes H(D))_n \to H_n(C\otimes D)$$

and

$$K : H_n(C\otimes D) \to \operatorname{Tor}_1(H(C), H(D))_{n-1}$$

that patch the two exact pieces. This yields the short exact sequence in degree n.

The map I is induced by $[c]\otimes[d] \mapsto [c\otimes d]$. This is Heller's proof. □

6 Dual theorem and universal coefficients

Definition 6.1. If $(C_\bullet, \varphi)$ and $(D_\bullet, \psi)$ are chain complexes, define the cochain complex $\operatorname{Hom}^\bullet(C, D)$ by

$$\operatorname{Hom}^n(C, D) = \prod_p \operatorname{Hom}(C_p, D_{p+n})$$

with differential

$$(df)_p = \psi f_p - (-1)^n f_{p-1}\varphi.$$

Lemma 6.2. *For $f \in \operatorname{Hom}^0(C, D)$ one has $df = 0$ if and only if f is a chain map. Also $f = dg$ for some $g \in \operatorname{Hom}^{-1}(C, D)$ if and only if f is null homotopic. Hence*

$$H^0 \operatorname{Hom}^\bullet(C, D) \cong \frac{\{\text{chain maps } C \to D\}}{\{\text{chain homotopies}\}}.$$

Proof. In degree 0, the identity $df = \psi f - f\varphi$ is exactly the chain map condition. In degree -1, the identity $f = dg = \psi g + g\varphi$ is exactly the chain homotopy formula. □

Theorem 6.3 (Dual form)**.** *Let $C_\bullet$ and $D_\bullet$ be bounded below chain complexes, with each C_n free abelian. Then for each n there is a natural short exact sequence*

$$0 \to \prod_p \operatorname{Ext}^1_{\mathbb{Z}}(H_p(C), H_{p+n-1}(D)) \to H^n \operatorname{Hom}^\bullet(C, D) \to \prod_p \operatorname{Hom}(H_p(C), H_{p+n}(D)) \to 0.$$

Proof. This is the formal dual of Theorem 5.7. Apply $\operatorname{Hom}^\bullet(-, D)$ to $0 \to B(C) \to Z(C) \to H(C) \to 0$ and compare the resulting cohomology exact sequence with the low-degree Ext sequence of Corollary 4.2. □

Corollary 6.4 (UCT for homology)**.** *Let $C_\bullet$ be a bounded below chain complex of free abelian groups and let G be an abelian group, regarded as a chain complex concentrated in degree 0. Then for each n there is a natural short exact sequence*

$$0 \to H_n(C) \otimes G \to H_n(C \otimes G) \to \operatorname{Tor}^{\mathbb{Z}}_1(H_{n-1}(C), G) \to 0.$$

Proof. Apply Theorem 5.7 to $C_\bullet$ and $G[0]$. □

Corollary 6.5 (UCT for cohomology)**.** *Let $C_\bullet$ be a bounded below chain complex of free abelian groups and let G be an abelian group. Then for each n there is a natural short exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(C), G) \to H^n \operatorname{Hom}^\bullet(C, G[0]) \to \operatorname{Hom}(H_n(C), G) \to 0.$$

Proof. Apply Theorem 6.3 to $D_\bullet = G[0]$. □

Remark 6.6. If G is flat over $\mathbb{Z}$, then the Tor term vanishes. If G is injective in **Ab**, for example if G is divisible, then the Ext term vanishes. The short exact sequences split as sequences of abelian groups, but not naturally in general.

7 Passage to topology

7.1 Singular chains and Eilenberg–Zilber

Definition 7.1. For a space X, let $C_n(X)$ be the free abelian group on singular n-simplices $\sigma : \Delta^n \to X$. The differential is

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \delta_i,$$

where $\delta_i : \Delta^{n-1} \to \Delta^n$ is the ith face map.

Definition 7.2. A (p, q)-shuffle is a permutation $\tau \in S_{p+q}$ such that

$$\tau(1) < \cdots < \tau(p) \qquad \text{and} \qquad \tau(p+1) < \cdots < \tau(p+q).$$

The set of (p, q)-shuffles is denoted $\mathrm{Sh}(p, q)$.

Definition 7.3. Let $\sigma : \Delta^n \to X \times Y$. Write $\sigma_X = \pi_X \sigma$ and $\sigma_Y = \pi_Y \sigma$. For $0 \le i_0 < \cdots < i_r \le n$, write

$$\sigma|[i_0, \ldots, i_r]$$

for the restriction of σ along the unique affine face map $\Delta^r \to \Delta^n$ onto the face spanned by $v_{i_0}, \ldots, v_{i_r}$.

Theorem 7.4 (Eilenberg–Zilber)**.** *For spaces X and Y there are natural chain maps*

$$\mathrm{AW} : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

and

$$\mathrm{Sh} : C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$$

which are chain homotopy inverse to each other.

The Alexander–Whitney map is given on a singular n-simplex $\sigma : \Delta^n \to X \times Y$ by

$$\mathrm{AW}(\sigma) = \sum_{p=0}^{n} \big(\sigma_X|[0, \ldots, p]\big) \otimes \big(\sigma_Y|[p, \ldots, n]\big).$$

For singular simplices $a : \Delta^p \to X$ and $b : \Delta^q \to Y$, the shuffle map is

$$\mathrm{Sh}(a \otimes b) = \sum_{\tau \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\tau)\big((a \circ \alpha_\tau), (b \circ \beta_\tau)\big),$$

where $\alpha_\tau : \Delta^{p+q} \to \Delta^p$ is the affine map sending the vertices

$$v_{\tau(1)}, \ldots, v_{\tau(p)}$$

to $v_1, \ldots, v_p$ in order and collapsing the remaining vertices in the unique order-preserving affine way, and $\beta_\tau : \Delta^{p+q} \to \Delta^q$ is defined similarly from

$$v_{\tau(p+1)}, \ldots, v_{\tau(p+q)}.$$

Thus each shuffle chooses which p vertices are sent to the X-simplex and which q vertices are sent to the Y-simplex.

Proof sketch. One checks directly that AW and Sh commute with the differentials. Then one constructs explicit natural chain homotopies between AWSh and the identity and between ShAW and the identity. Hence the two complexes are chain homotopy equivalent. See Hatcher, Section 3.B. □

Corollary 7.5. *For all spaces X and Y,*

$$H_n\big(C_\bullet(X) \otimes C_\bullet(Y)\big) \cong H_n(X \times Y; \mathbb{Z}).$$

Proof. Apply homology to the chain homotopy equivalence of Theorem 7.4. □

7.2 Künneth and UCT for spaces

Theorem 7.6 (Topological Künneth theorem)**.** *For spaces X and Y, and each $n \geq 0$, there is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes H_q(Y;\mathbb{Z}) \to H_n(X \times Y;\mathbb{Z}) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^{\mathbb{Z}}\big(H_p(X;\mathbb{Z}), H_q(Y;\mathbb{Z})\big) \to 0.$$

Proof. Each $C_n(X)$ and $C_n(Y)$ is free abelian. Apply Theorem 5.7 to $C_\bullet(X)$ and $C_\bullet(Y)$, then use Theorem 7.4. □

Corollary 7.7 (UCT for homology)**.** *For every space X, every abelian group G, and every $n \geq 0$, there is a natural short exact sequence*

$$0 \to H_n(X;\mathbb{Z}) \otimes G \to H_n(X;G) \to \operatorname{Tor}_1^{\mathbb{Z}}\big(H_{n-1}(X;\mathbb{Z}), G\big) \to 0.$$

Proof. Apply Corollary 6.4 to the singular chain complex $C_\bullet(X)$. □

Corollary 7.8 (UCT for cohomology)**.** *For every space X, every abelian group G, and every $n \geq 0$, there is a natural short exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}\big(H_{n-1}(X;\mathbb{Z}), G\big) \to H^n(X;G) \to \operatorname{Hom}\big(H_n(X;\mathbb{Z}), G\big) \to 0.$$

Proof. Apply Corollary 6.5 to the singular chain complex $C_\bullet(X)$. □

7.3 Examples

Example 7.9 (The torus)**.** Let $T^2 = S^1 \times S^1$. Since

$$H_i(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i \geq 2 \end{cases}$$

and all groups are free, the Tor term vanishes. Hence

$$H_0(T^2;\mathbb{Z}) \cong \mathbb{Z}, \qquad H_1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2, \qquad H_2(T^2;\mathbb{Z}) \cong \mathbb{Z}.$$

Also $H_n(T^2;\mathbb{Z}) = 0$ for $n \geq 3$.

Example 7.10 (Products of spheres)**.** If $m < n$, then

$$H_k(S^m \times S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, m, n, m+n \\ 0 & \text{otherwise} \end{cases}$$

because sphere homology is free. If $m = n$, then

$$H_k(S^n \times S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z}^2 & k = n \\ 0 & \text{otherwise} \end{cases}$$

Example 7.11 (The three-torus)**.** Apply Künneth twice. One gets

$$H_k(T^3;\mathbb{Z}) \cong \mathbb{Z}^{\binom{3}{k}} \qquad 0 \le k \le 3.$$

In particular

$$H_0(T^3;\mathbb{Z}) \cong \mathbb{Z}, \qquad H_1(T^3;\mathbb{Z}) \cong \mathbb{Z}^3, \qquad H_2(T^3;\mathbb{Z}) \cong \mathbb{Z}^3, \qquad H_3(T^3;\mathbb{Z}) \cong \mathbb{Z}.$$

Example 7.12 (A first Tor term from topology)**.** Take $X = \mathbb{RP}^2$ and coefficients $\mathbb{Z}/2$. Since

$$H_0(X;\mathbb{Z}) \cong \mathbb{Z}, \qquad H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2, \qquad H_i(X;\mathbb{Z}) = 0 \text{ for } i \ge 2,$$

the homology UCT gives

$$0 \to H_2(X;\mathbb{Z}) \otimes \mathbb{Z}/2 \to H_2(X;\mathbb{Z}/2) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2) \to 0.$$

Since $H_2(X;\mathbb{Z}) = 0$ and $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2$, we obtain

$$H_2(\mathbb{RP}^2;\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

So a new class appears only after changing coefficients.

Example 7.13 (A first Ext term from topology)**.** Again let $X = \mathbb{RP}^2$. The cohomology UCT in degree 2 gives

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Now $H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2$, $H_2(X;\mathbb{Z}) = 0$, and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2$. Hence

$$H^2(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}/2.$$

This group is entirely an Ext term.

Example 7.14 (A Künneth computation with nonzero Tor)**.** Take $X = \mathbb{RP}^3$ and $Y = \mathbb{RP}^2$. Their integral homology groups are

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}/2 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_j(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}/2 & j = 1 \\ 0 & \text{otherwise} \end{cases}$$

In degree 3, Künneth gives

$$0 \to H_3(X;\mathbb{Z}) \otimes H_0(Y;\mathbb{Z}) \to H_3(X \times Y;\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_1(X;\mathbb{Z}), H_1(Y;\mathbb{Z})) \to 0.$$

Thus

$$0 \to \mathbb{Z} \to H_3(\mathbb{RP}^3 \times \mathbb{RP}^2;\mathbb{Z}) \to \mathbb{Z}/2 \to 0.$$

Since the sequence splits as a sequence of abelian groups,

$$H_3(\mathbb{RP}^3 \times \mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

Example 7.15 (Coefficients on the torus)**.** Let $G = \mathbb{Z}/n$. Since the integral homology of T^2 is free, the homology UCT gives

$$H_k(T^2; G) \cong H_k(T^2; \mathbb{Z}) \otimes G.$$

Hence

$$H_0(T^2; \mathbb{Z}/n) \cong \mathbb{Z}/n, \qquad H_1(T^2; \mathbb{Z}/n) \cong (\mathbb{Z}/n)^2, \qquad H_2(T^2; \mathbb{Z}/n) \cong \mathbb{Z}/n.$$

Example 7.16 (Lens spaces)**.** Let $L = L(p, q)$ be a lens space. Its integral homology is

$$H_0(L; \mathbb{Z}) \cong \mathbb{Z}, \qquad H_1(L; \mathbb{Z}) \cong \mathbb{Z}/p, \qquad H_2(L; \mathbb{Z}) = 0, \qquad H_3(L; \mathbb{Z}) \cong \mathbb{Z}.$$

The cohomology UCT yields

$$H^2(L; \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}) \cong \mathbb{Z}/p,$$

and

$$H^1(L; \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}) = 0.$$

So the degree 2 integral cohomology of a lens space is an Ext group.

Example 7.17 (Changing coefficients can erase torsion)**.** Let $X = \mathbb{RP}^2 \times \mathbb{RP}^2$. Over $\mathbb{Q}$ one has

$$H_i(\mathbb{RP}^2; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ 0 & i \geq 1 \end{cases}$$

so Künneth gives

$$H_i(X; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ 0 & i \geq 1. \end{cases}$$

Over $\mathbb{F}_2$ one has

$$H_i(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2 \qquad \text{for } i = 0, 1, 2,$$

hence

$$H_n(X; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & n = 0, 4 \\ \mathbb{F}_2^2 & n = 1, 3 \\ \mathbb{F}_2^3 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 7.18. The examples show three recurrent patterns.

(i) If integral homology is free, Künneth reduces to tensor products.

(ii) Torsion in integral homology produces Tor and Ext terms.

(iii) Changing coefficients can create classes or erase them.

References

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